The Bohemian Eigenvalue Project

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Introduction

Bohemian eigenvalues are the eigenvalues of matrices with entries of bounded height, typically drawn from a discrete set. The name “Bohemian” is intended as a mnemonic and is derived from “bounded height integer matrices.” These objects are surprisingly interesting to study, with many unsolved problems related to them.

Bohemian eigenvalue problems are a general extension of the analogous polynomial problem [1 – 5] because every bounded height polynomial is a structured Bohemian eigenvalue problem (just use the Frobenius companion matrix).

Many of the visible features remain unexplained, such as the eigenvalue exclusion regions (regions where no eigenvalues appear, see Figure 2), diffraction patterns (see Figures 1 and 3), and star shaped boundaries around some exclusion regions (see Figure 2 – bottom right).

See the works of Tao and Vu [7,8] for universality results for larger dimension in the generic structured case. This project concentrates on explicit construction of high resolution images of the eigenvalues for modest dimensions and sizes of the entries.

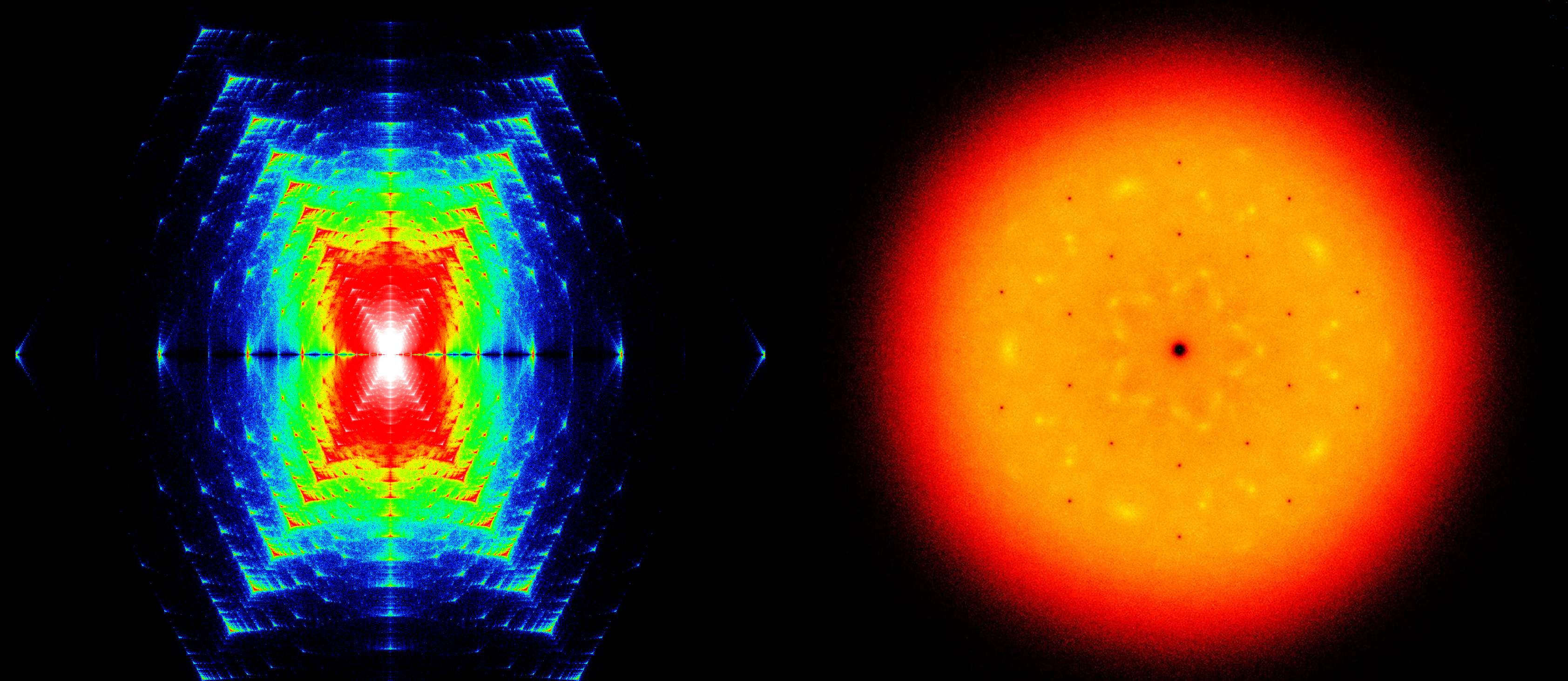

Figure 1: Two images of Bohemian eigenvalues from different classes of matrices plotted on the complex plane and colored by eigenvalue density. **Left:** A sample of 50 million matrices from the set of 5×5 matrices where the entries are sampled from the set of Fibonacci numbers up to 514,229. This image is viewed on $-550{,}000 - 550{,}000i$ to $550{,}000 + 550{,}000i$. **Right:** A sample of 20 million matrices from the set of 5×5 matrices where the entries are sampled from the set $\{e^{2k\pi i/5}, k = 0 \ldots 4\}$. This image is viewed on $-4 - 4i$ to $4 + 4i$.

References

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[5] John E. Littlewood. On polynomials $\sum^n \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta_i}$. *Journal of the London Mathematical Society*, 41:367–376, 1966.

[6] Neil J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. Sequences A272658–A272663, A271570, A271587, A271588.

[7] Terence Tao and Van Vu. Random matrices: the circular law. *Communications in Contemporary Mathematics*, 10(02):261–307, 2008.

[8] Terence Tao and Van Vu. Random matrices: universality of local eigenvalue statistics. *Acta mathematica*, 206(1):127–204, 2011.

Figure 4 (Bottom): Each of these 14 images come a class of 4×4 matrices where all entries are fixed integers with the exception of two parametric entries. Each image uses a different base matrix (the non-parametric entries in the matrix) where the base entries are integers between -2 and 2. The eigenvalues from a sample of 1 million matrices, where the parametric entries are drawn from a continuous uniform distribution on the interval $[-2, 2]$, is shown.

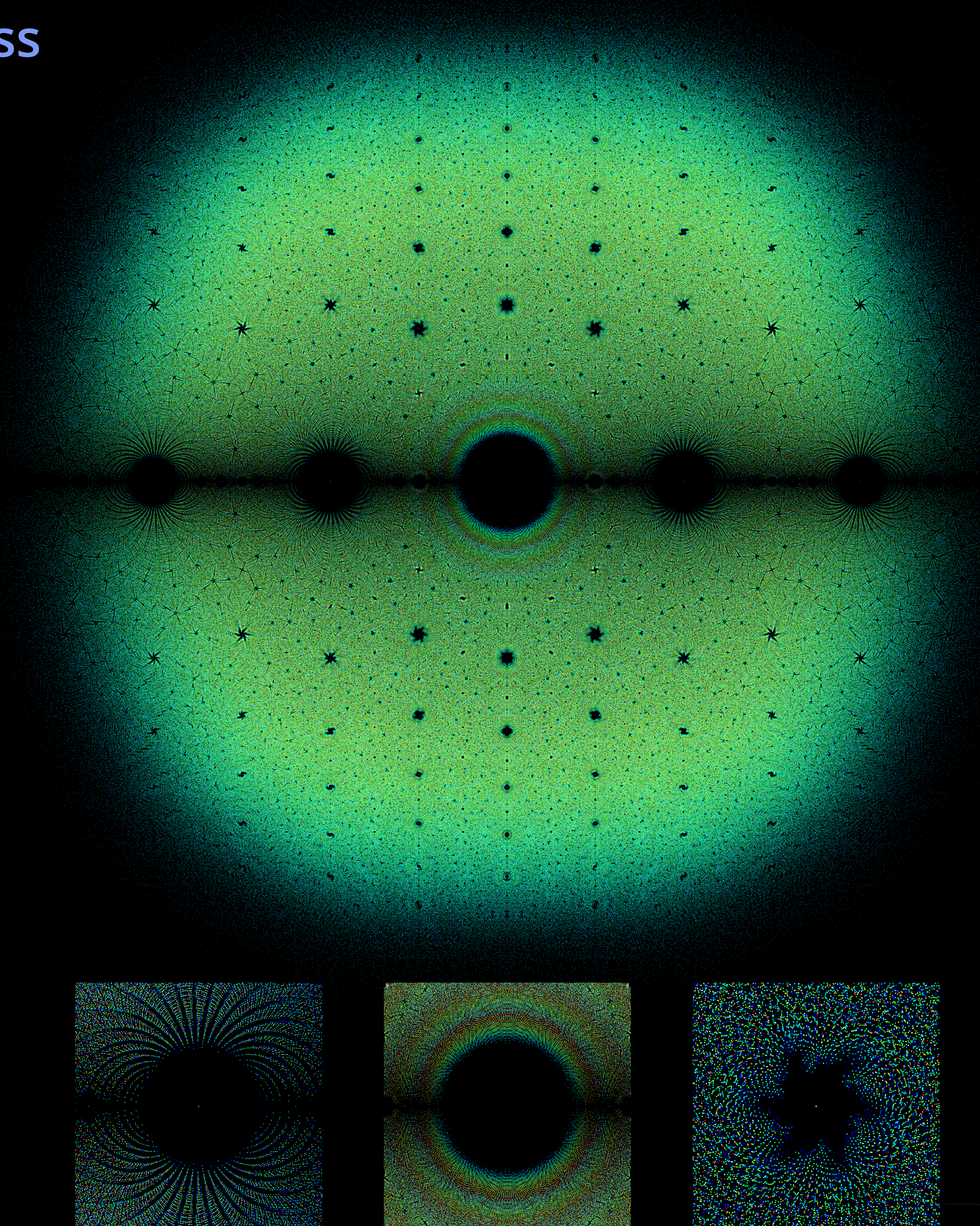

Figure 2: Plot on the complex plane of all the eigenvalues from the set of 5×5 matrices with entries in $\{-1, 0, 1\}$. This class of matrices contains a total of $3^{5^2} \approx 8.5 \times 10^{11}$ matrices. Real eigenvalues have been omitted from this plot. Coloring represents the density of the eigenvalues. **Top:** Viewed on $-3.3 - 3.3i$ to $3.3 + 3.3i$. **Bottom Left:** The exclusion region surrounding the eigenvalue at $+2$. **Bottom Center:** The exclusion region surrounding the origin. **Bottom Right:** The exclusion region surrounding the eigenvalue at $1/2 + i\sqrt{3}/2$.

Matrix Size	Number of Matrices	Number of Characteristic Polynomials	Number of Eigenvalues	Number of Distinct Minimal Polynomials	Number of Matrices with Multiple Eigenvalues
OEIS	A060722	A272658	A271570	A271587	A271588
1×1	$3^1 = 3$	3	3	3	0
2×2	$3^{2^2} = 81$	16	21	19	19
3×3	$3^{3^2} = 19,683$	209	375	220	4,629
4×4	$3^{4^2} = 43,046,721$	8,739	24,823	8,924	7,171,257
5×5	$3^{5^2} = 847,288,609,443$	1,839,102	?	?	89,765,448,427
6×6	$3^{6^2} = 150,094,635,296,999,121$	?	?	?	?

Table 1: Properties of the class of $n \times n$ matrices with entries from $\{-1, 0, 1\}$. Sequence numbers are given for The Online Encyclopedia of Integer Sequences [6].

Computation

Due to the combinatorial growth of classes of matrices, an exhaustive computation of all eigenvalues is often infeasible. Instead, a large sample of matrices is used (often 10 million to over 1 billion). This gives a sufficient representation of the features visible in the resulting images.

Many of the images on this poster were created using a Matlab script utilizing the Parallel Computing Toolbox and vectorization. As a benchmark, this script is capable of computing and storing the eigenvalues of approximately 10 billion 5×5 matrices in 24 hours* using 8 parallel processes.

While computing a random sample can be useful for visualization, it cannot be used for accurately determining properties of classes of matrices (see Table 1). For some reasonably sized classes of matrices, we can compute and store all characteristic polynomials. This approach is useful when a class of matrices has less than about 10 trillion matrices (on a Desktop computer), with some additional restrictions on the number of possible values for the entries and the size of the matrices (related to memory usage).

For classes of matrices with less than 100 million entries, Maple can handle the computations in a reasonable amount of time (1×1 to 4×4 properties in Table 1 were computed this way). Beyond this, OpenMP with C++ was used to determine the set of all characteristic polynomials for the class of 5×5 matrices with entries drawn from $\{-1, 0, 1\}$ (see Figure 2). The complete distribution for this class (characteristic polynomials and their densities) was computed in under 6 hours* using 8 parallel processes (847 billion matrices!) If we had tried to do this using the Matlab script we would have to wait roughly 3 months* before we had the full distribution of eigenvalues. Not to mention, this would take more than 15TB of storage space (storing each eigenvalue in single precision) and, due to numerical errors, would not be useful in determining properties of this class of matrices.

Analysing classes of matrices that are larger than those explored here would require a deeper understanding of the symmetries that are present in these classes of matrices. The 6×6 row in Table 1 can be reduced by a factor of 28 through the use of row and column permutations for similar matrices, though this is still far from enough to be computationally feasible.

* A 2016, 3.3GHz quad-core Intel Core i7 iMac with 16GB RAM and 256GB flash storage was used for all computations. Any timings refer to this computer.

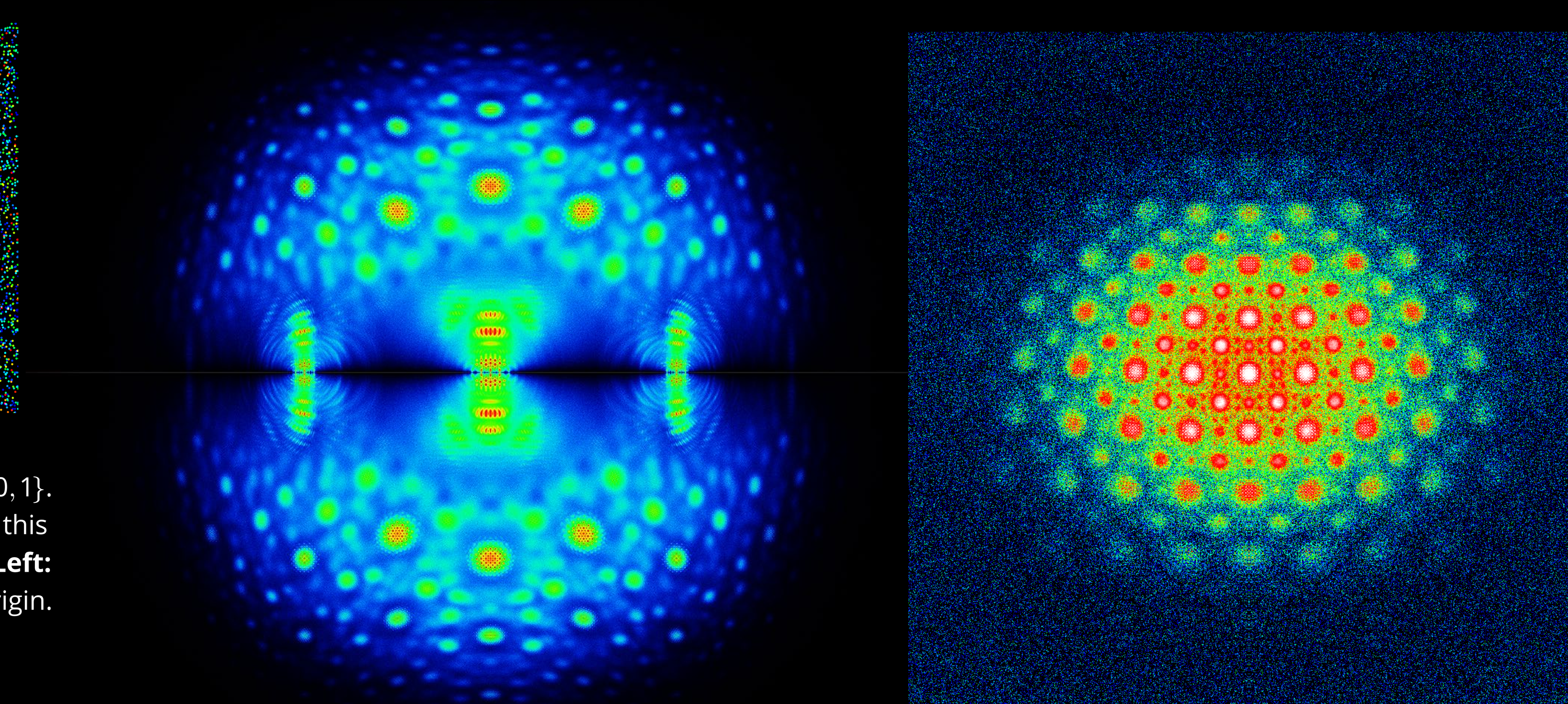

Figure 3: Plot on the complex plane of the eigenvalues from a sample of 73 million 5×5 matrices with entries in $\{-20, -1, 0, 1, 20\}$. Real eigenvalues have been omitted from this plot. Coloring represents the density of the eigenvalues. **Left:** Viewed on $-50 - 40i$ to $50 + 40i$. **Right:** The diffraction pattern centered at $+20i$.

Source code with examples available at:
github.com/StevenThornton/BHIME-Project

Acknowledgments

We would like to thank Daniel Lichtblau for verifying our findings for the number of unique characteristic polynomials for several classes of matrices, as well as for his suggestions for improving our method of counting characteristic polynomials. We also thank Erich Kaltofen for pointing out references to Tao and Vu for the asymptotic distributions of these problems. Finally, we would like to thank Sonia Gupta, Jonathan Brino-Tarasoff and Venkat Balasubramanian for their contributions to this project.