Visualizing Eigenvalues of Structured Random Matrices

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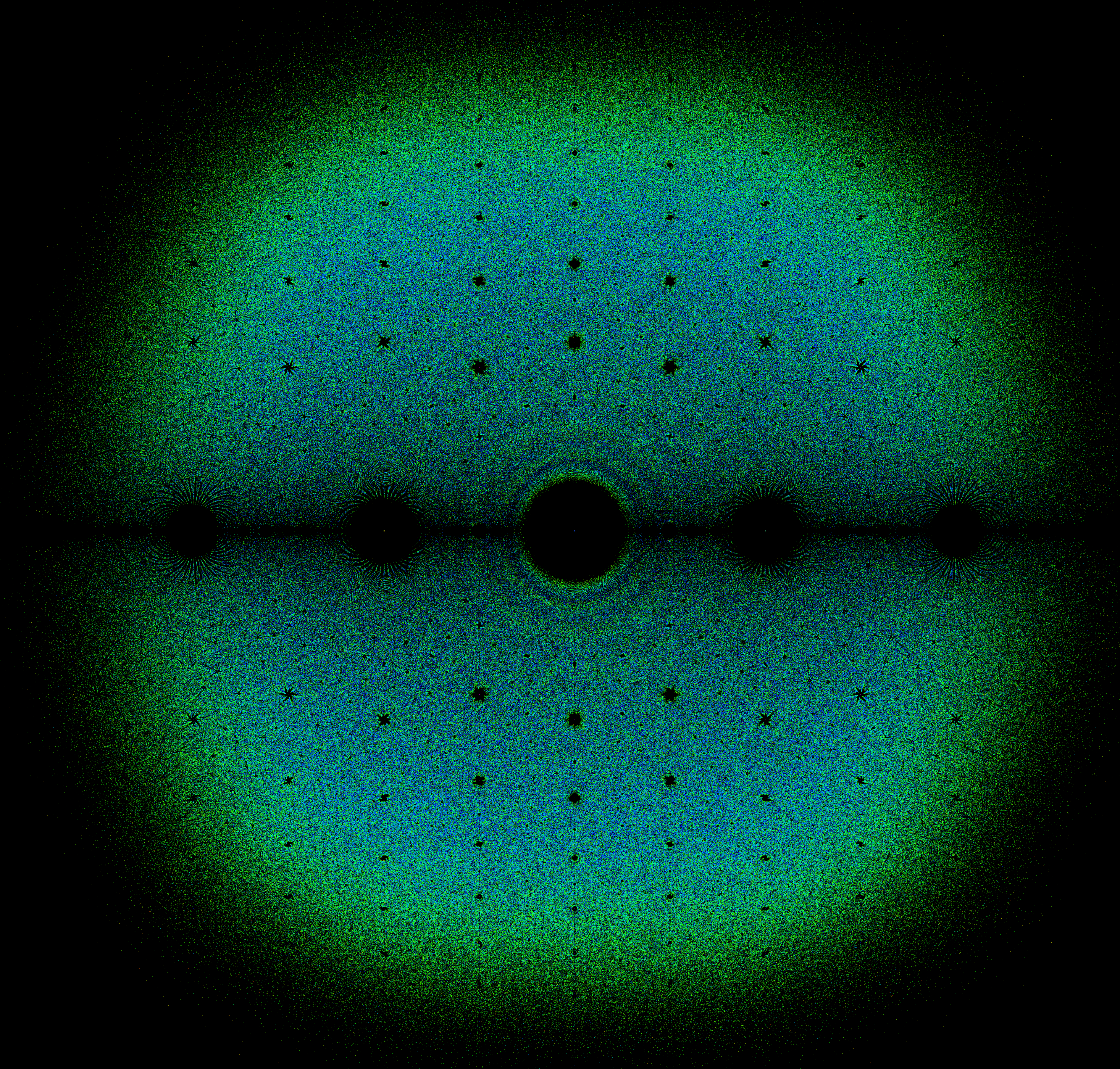

Figure 1: Plot on the complex plane of the eigenvalues from a sample of 3×10^9 matrices from the set of 5×5 matrices with entries in $\{-1, 0, 1\}$. This class of matrices contains a total of $3^{5^2} \approx 8.5 \times 10^{11}$ elements. The plot is viewed from $-3.3 - 3.3i$ to $3.3 + 3.3i$. Symmetry across the real and imaginary axes is used to give an effective 6×10^{10} points. Coloring is based on the density at each point where the highest density points are purple and the lowest density is yellow.

Bounded Integer Matrices

The set of $n \times n$ matrices with integer coefficients ranging from $-p$ to p contains p^{n^2} elements. The eigenvalues of these matrices remain bounded within a circle in the complex plane of radius pn. When these eigenvalues are plotted on the complex plane, some interesting structure arises. Figure 1 shows a plot of a sample for 5×5 matrices with coefficients in $\{-1, 0, 1\}$.

Regions containing no eigenvalues except for at the center appear, we call these *eigenvalue exclusion regions*. The exclusion regions are clearly visible in figure 1. They are largest around integers along the real axis. We are working on determining a bound for the size of the exclusion regions. Specifically, we are looking for a bound on the size of the exclusion region centered at the origin (figure 2). As a starting point we are interested in the size of the exclusion region on the real axis. We are looking at the correlation between the coefficients of the linear portion of the characteristic polynomials to determine a bound. For the bound in the complex plane we are looking into the coefficients of the quadratic portion of the characteristic polynomials.

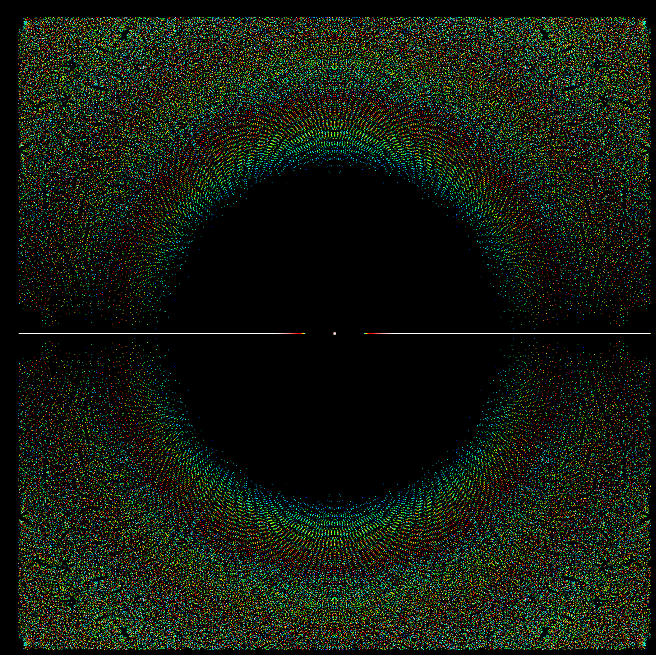

Figure 2: The plot from figure 1 viewed on $-0.5 - 0.5i$ to $0.5 + 0.5i$. The eigenvalues along the real axis are visible and a a clear gap can be seen between the smallest real eigenvalues and the origin. Other points close to the origin are

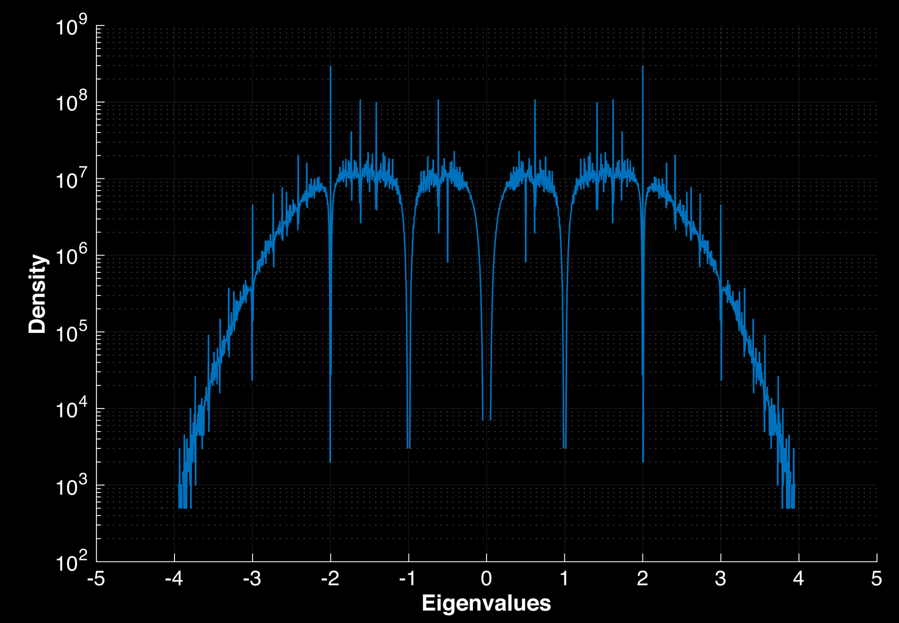

Figure 3: Distribution of the real eigenvalues from figure 1 on a logarithmic scale.

Computation

Matlab was used to compute the eigenvalues for samples of matrices from various classes. It was chosen as it provides fast eigenvalue computation as well as the parallel computing toolbox to maximize computing power.

Computing the raw data: Matrices are sampled from the appropriate distribution and the eigenvalues and their respective condition numbers are computed using Matlab's `condeig` function. Data is saved to .mat files and broken into chunks no larger than 500mb to avoid using more memory than is available.

Processing the data: The data was processed into a grid of points where each point represents a pixel in the output image. If coloring is by density, then a grid of integers is used and the number of eigenvalues falling in each point is counted. If coloring is by condition number, then each grid point is the average eigenvalue condition number for the eigenvalues that fall within that point.

Making an image: The final output plot is made using Matlab. The logarithm of the raw data for either the eigenvalue density or average condition number is plotted on a heat map with the appropriate colormap applied.

Beta Distribution

Interesting properties of classes of random matrices appear when the eigenvalues are computed for matrices where the entries follow a shifted and scaled Beta distribution (figure 1). Figures 2 and 3 show plots on the complex plane of the eigenvalues for large samples of matrices with entries following a distribution $2X_{i,j} - 1$ for $X_{i,j} \sim \mathrm{Beta}(\alpha = 0.01, \beta = 0.01)$ for 3×3 and 4×4 matrices respectively.

The nodes visible on the plots (most visible in figure 3) are the set of eigenvalues of matrices whose entries are 1 or -1. The curves connecting these nodes are algebraic curves arising from the roots of the characteristic polynomial of a matrix whose entries are $-1, 1$ with the exception of one entries that varies from -1 to 1.

The coloring in figure 2 shows the average eigenvalue condition number for the data computed. Clearly as eigenvalues approach the real axis (especially near the origin) the condition number increases (as is shown in pink).

The coloring in figure 3 shows the density at each point White indicates the highest density, typically at the nodes, and blue is the lowest density.

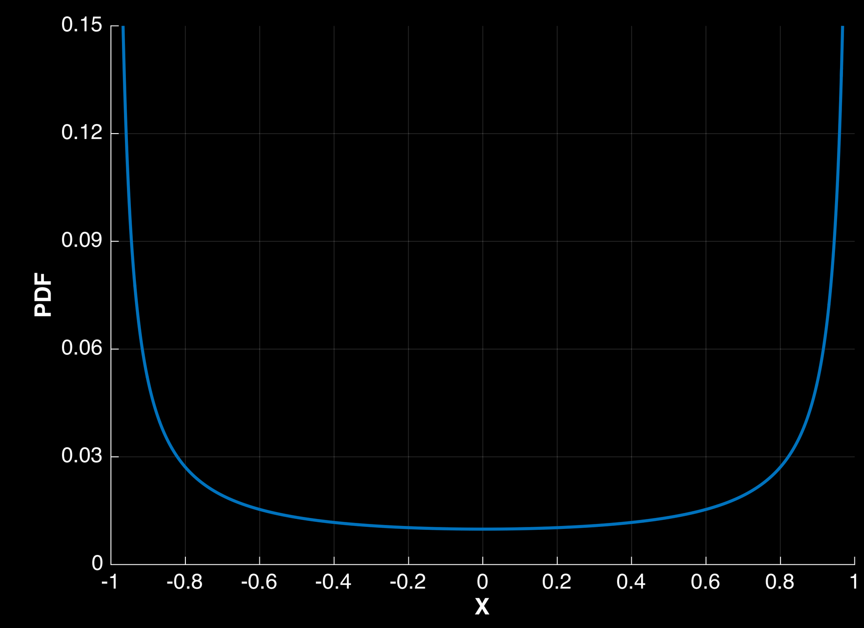

Figure 4: Probability distribution function for a random variable X where $X = 2Y - 1$ and $Y \sim \mathrm{Beta}(\alpha = 0.01, \beta = 0.01)$. That is, X is a shifted and scaled Beta random variable where the probability of being near 1 or -1 is high and the probability of being near zero is small.

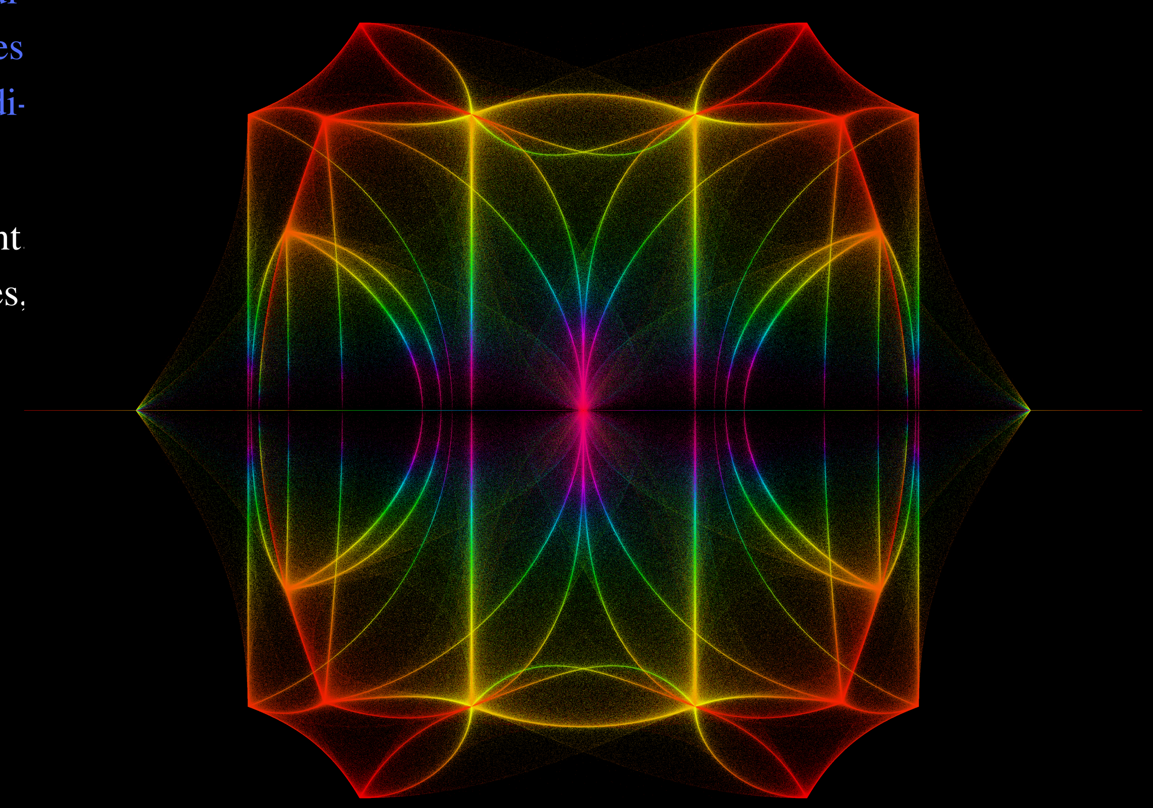

Figure 5: Eigenvalues of 6×10^6 randomly sampled 3×3 matrices with entries $2X_{i,j} - 1$ for $X_{i,j} \sim \mathrm{Beta}(\alpha = 0.01, \beta = 0.01)$, $X_{i,j}$ i.i.d. The plot is on the complex plane viewed from $-2.5 - 2.5i$ to $2.5 + 2.5i$]. Coloring is based on the log of the average eigenvalue condition number where the minimum is 1 (colored red) and the maximum is 1.27×10^4 (colored pink).

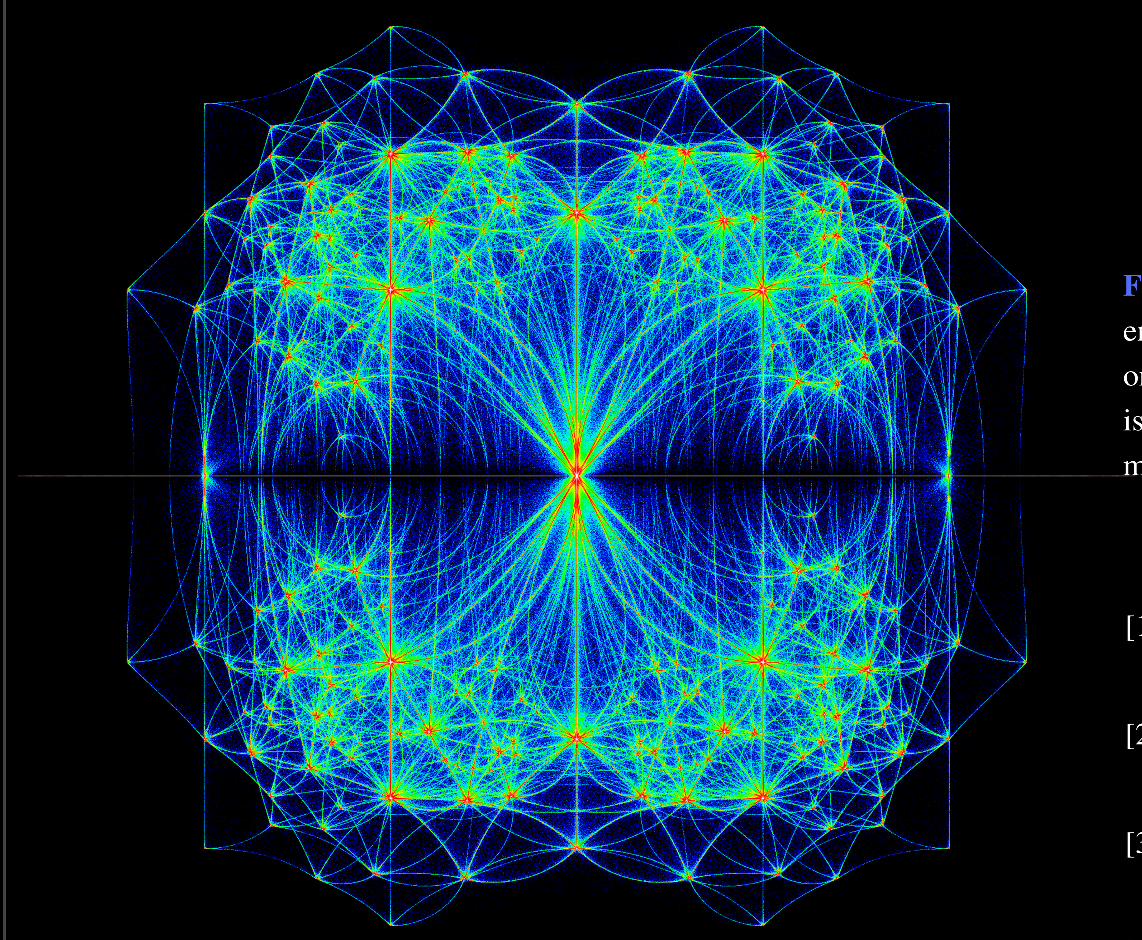

Figure 6: Eigenvalues of 10^6 randomly sampled 4×4 matrices with entries $2X_{i,j} - 1$ for $X_{i,j} \sim \mathrm{Beta}(\alpha = 0.01, \beta = 0.01)$, $X_{i,j}$ i.i.d. The plot is on the complex plane viewed from $-3 - 3i$ to $3 + 3i$. Coloring indicates the density of eigenvalues where blue is the least dense and white is the highest density.

References

[1] Peter Borwein and Loki Jörgenson. Visible structures in number theory. *American Mathematical Monthly*, pages 897–910, 2001.

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[3] Dan Christensen. Plots of roots of polynomials with integer coefficients. `http://jdc.math.uwo.ca/roots/`. Accessed: 2015-10-01.

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Figure 7: (bottom) Each of these 14 images is a complex plot of the eigenvalues of a sample of matrices where all elements are fixed integers except for two elements that follow a (continuous) uniform distribution.