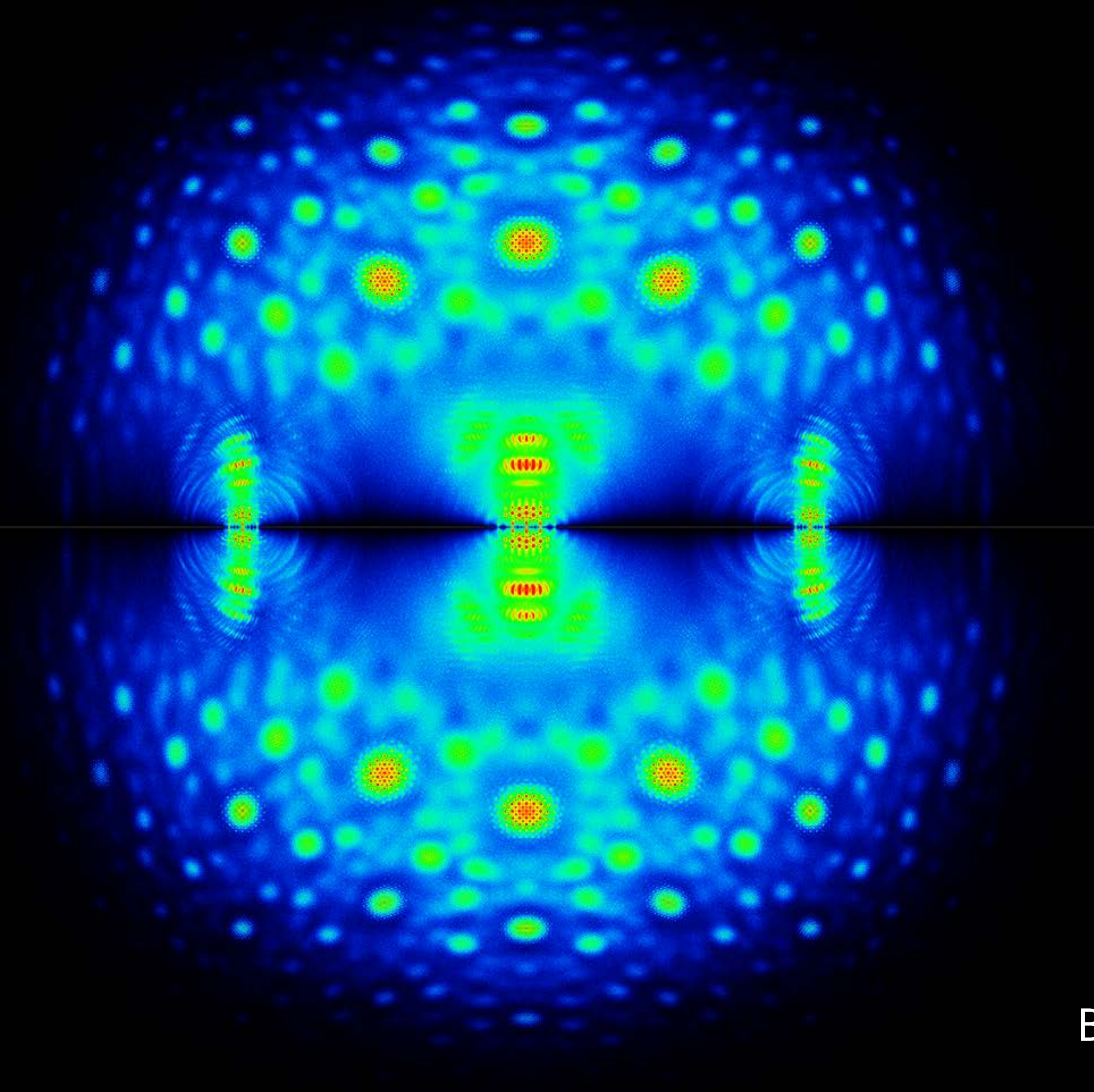
Bohemian Eigenvalues



Steven E. Thornton
Robert M. Corless

BohemianEigenvalues.com

Definitions

Bounded Height Integer Matrix Eigenvalues $=$ **BHIME**

Family of **Bohemian Matrices**

- Entries sampled from a fixed discrete set of bounded height

Bohemian Eigenvalues

- The eigenvalues of Bohemian matrices

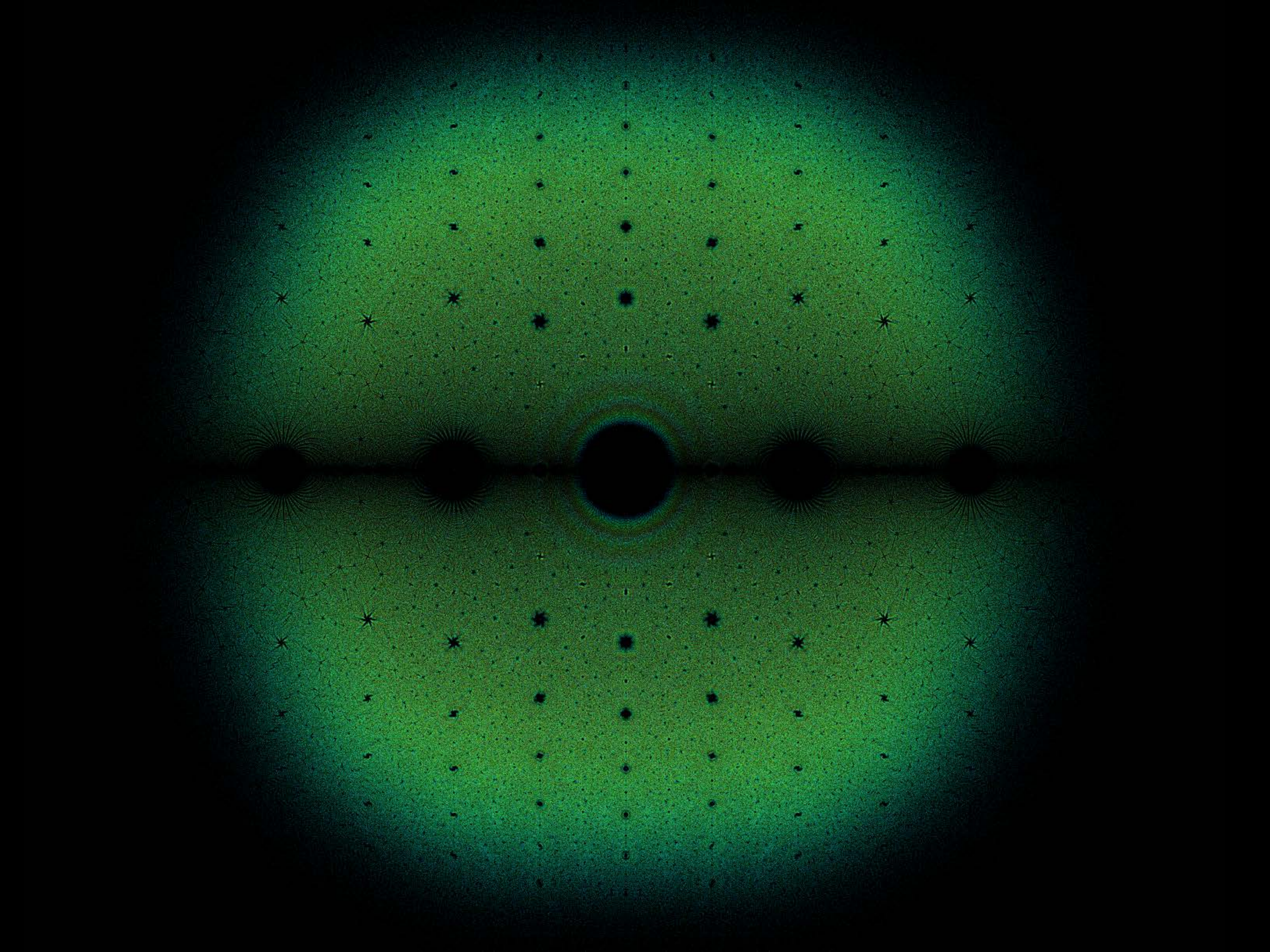
Rhapsody

- A Bohemian matrix whose inverse belongs to the same family of Bohemian matrices.

“Number theory is the queen of mathematics.”
— Gauss

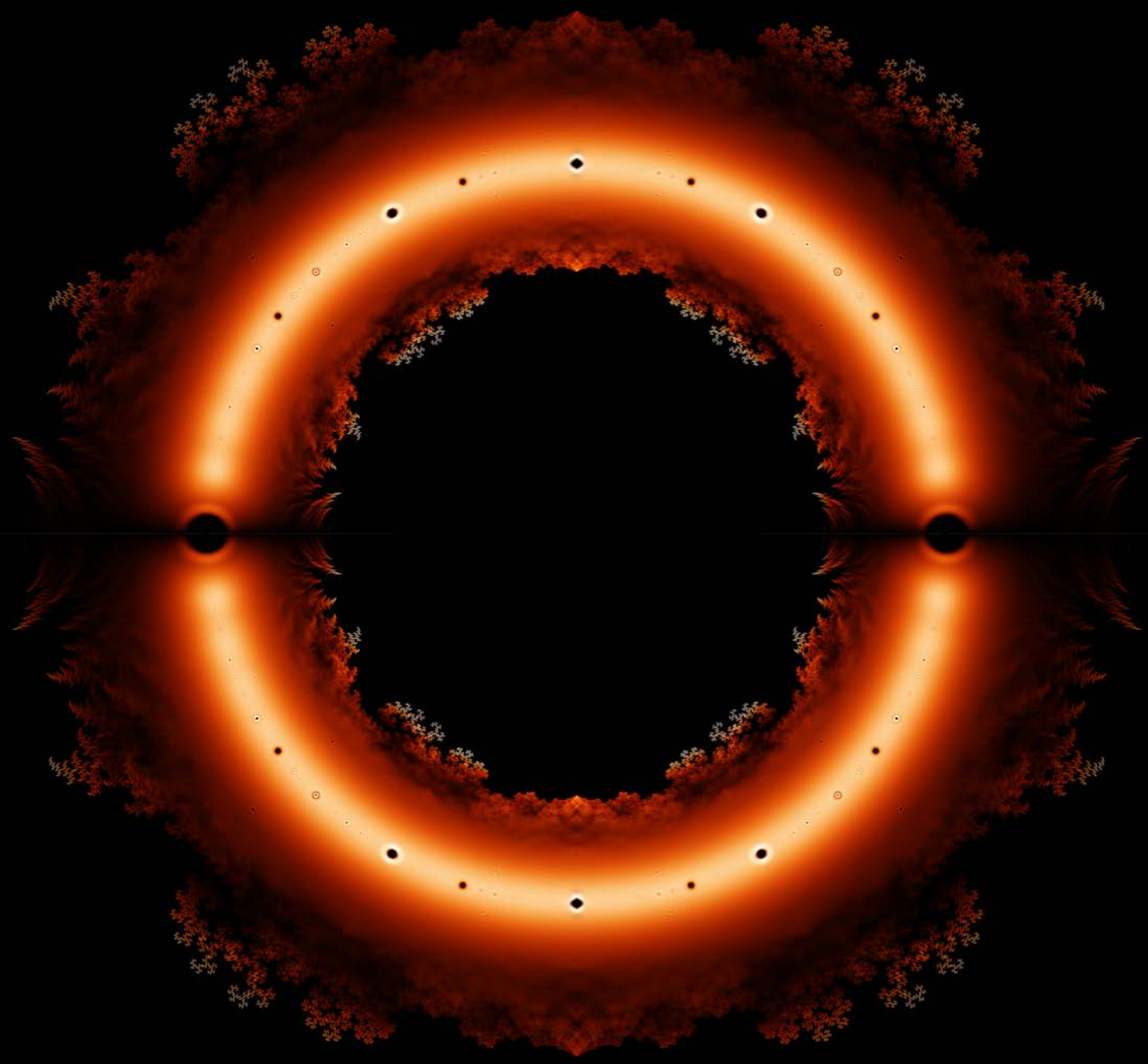
Example

5×5 matrices where the entries are uniformly sampled from the set $\{-1, 0, 1\}$.



History

- Polynomials with coefficients of bounded height: Littlewood (50s), P. Borwein & L. Jörgenson 1995, D. Christensen, J. Baez more recently
- Corless 2004/2007 Lagrange basis pictures, and the realization that PBH $\subseteq$ Bohemian matrices because of companion matrices
- Lawrence & Corless 2011, Mandelbrot matrices show PBH $\subset$ Bohemian ($F = \{0, 1\}$ but coefficients of p grow exponentially in degree)
- Random matrices: Wishart 1928, Wigner 1967, Tao & Vu 2009, 2015
- Graph theory (incidence matrices)
- Finite difference matrices



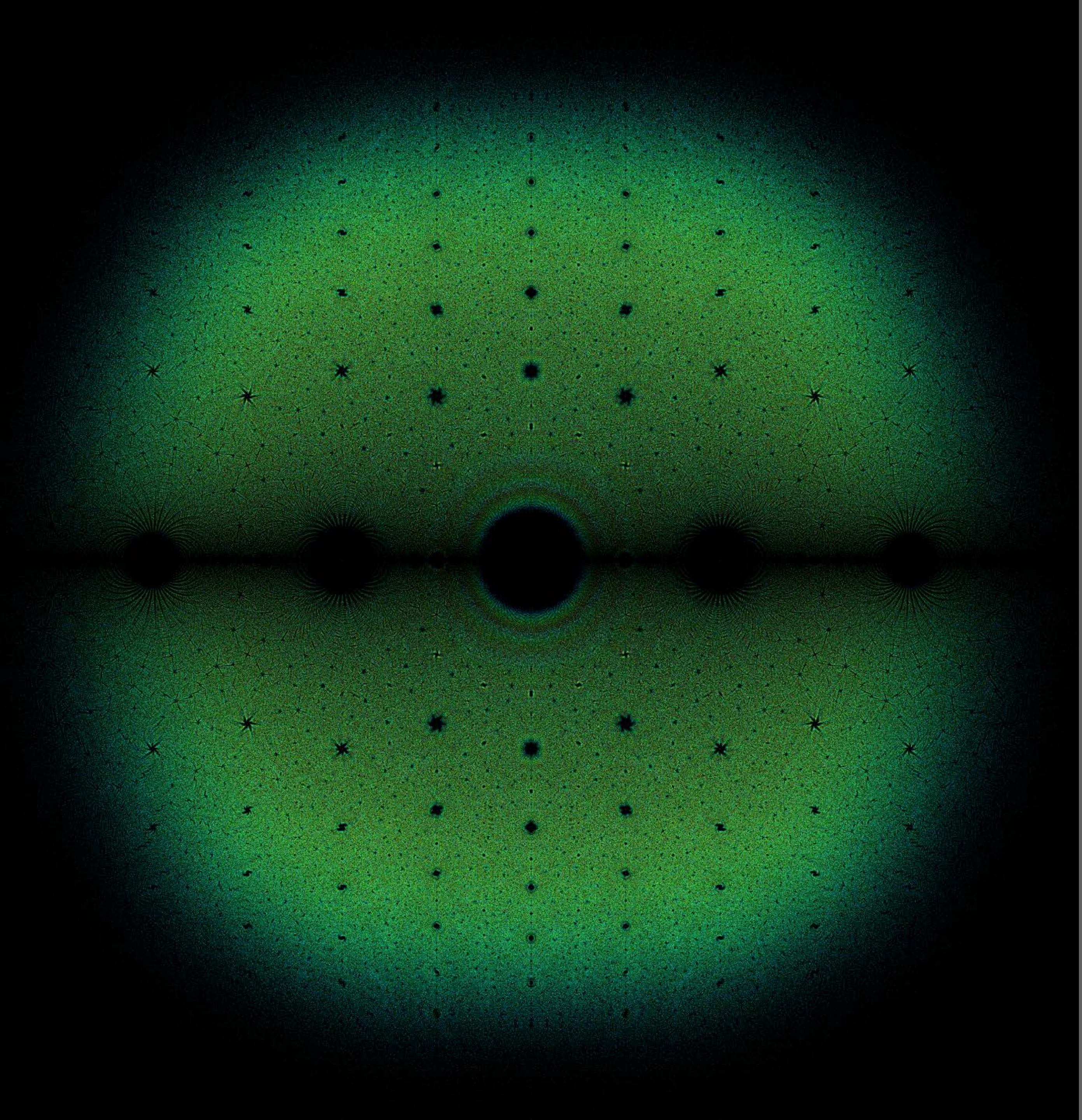
Source: `math.ucr.edu/home/baez/roots/`

Details

- Roots of all polynomials of degree 24
- Coefficients in $\{-1, 1\}$
- Created by Sam Derbyshire
- 4 days to compute roots using Mathematica

Eigenvalues of Random Matrices

- Edelman, A. (1988). Eigenvalues and condition numbers of random matrices. *SIAM Journal on Matrix Analysis and Applications*, 9(4), 543-560.
- Marčenko, V. A., & Pastur, L. A. (1967). Distribution of Eigenvalues for Some Sets of Random Matrices. *Mathematics of the USSR-Sbornik*, 1(4), 457.
- Tao, T., & Vu, V. (2011). Random Matrices: Universality of Local Eigenvalue Statistics. *Acta mathematica*, 206(1), 127-204.
- Diaconis, P., & Shahshahani, M. (1994). On the Eigenvalues of Random Matrices. *Journal of Applied Probability*, 49-62.
- Arnold, L. (1967). On the Asymptotic Distribution of the Eigenvalues of Random Matrices. *Journal of Mathematical Analysis and Applications*, 20(2), 262-268.



Details

- 5×5 matrices
- Entries in the set $\{-1, 0, 1\}$
- 7,963,249 distinct eigenvalues
- Approximately 6 hours to produce this image (on a 2016 MacBook Pro, 2.8GHz Intel Core i7, 16GB RAM)

Some Properties

Consider the class of $n \times n$ matrices where the entries are from the set $\{-1, 0, 1\}$.

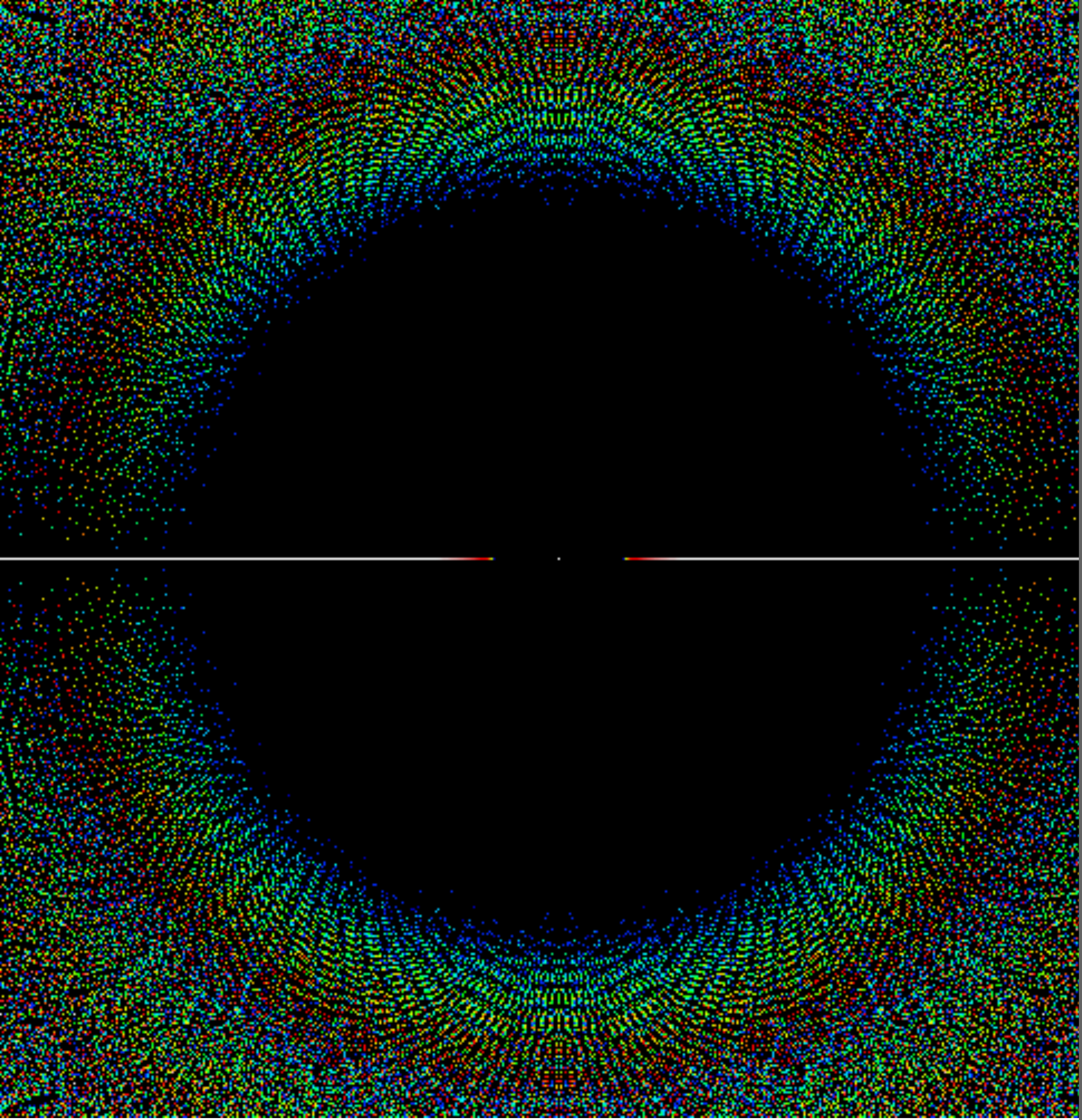
Matrix Size	Number of Matrices	Number of Distinct Characteristic Polynomials	Number of Distinct Minimal Polynomials	Number of Distinct Eigenvalues	Number of Nonsingular Matrices
OEIS	A060722	A272658	A271587	A271570	A056989
1×1	$3^{1^2} = 3$	3	3	3	2
2×2	$3^{2^2} = 81$	16	19	21	48
3×3	$3^{3^2} = 19{,}683$	209	220	375	11,808
4×4	$3^{4^2} = 43{,}046{,}721$	8,739	8,924	24,832	27,974,420
5×5	$3^{5^2} = 847{,}288{,}609{,}443$	1,839,102	?	7,963,249	609,653,621,760
6×6	3^{6^2}	?	?	?	?

Duplicate Eigenvalues

- For $F = \{0, 1\}$, $n = 3$ there are $2^{3^2} = 512$ matrices but only 32 distinct characteristic polynomials
- $\lambda(\lambda - 1)^2$ and $\lambda^2(\lambda - 1)$ occur 75 times each
- $(\lambda - 2)(\lambda + 1)^2$, $\lambda^2(\lambda - 3)$ **once** each
- Possible that some eigenvalues are more likely than others

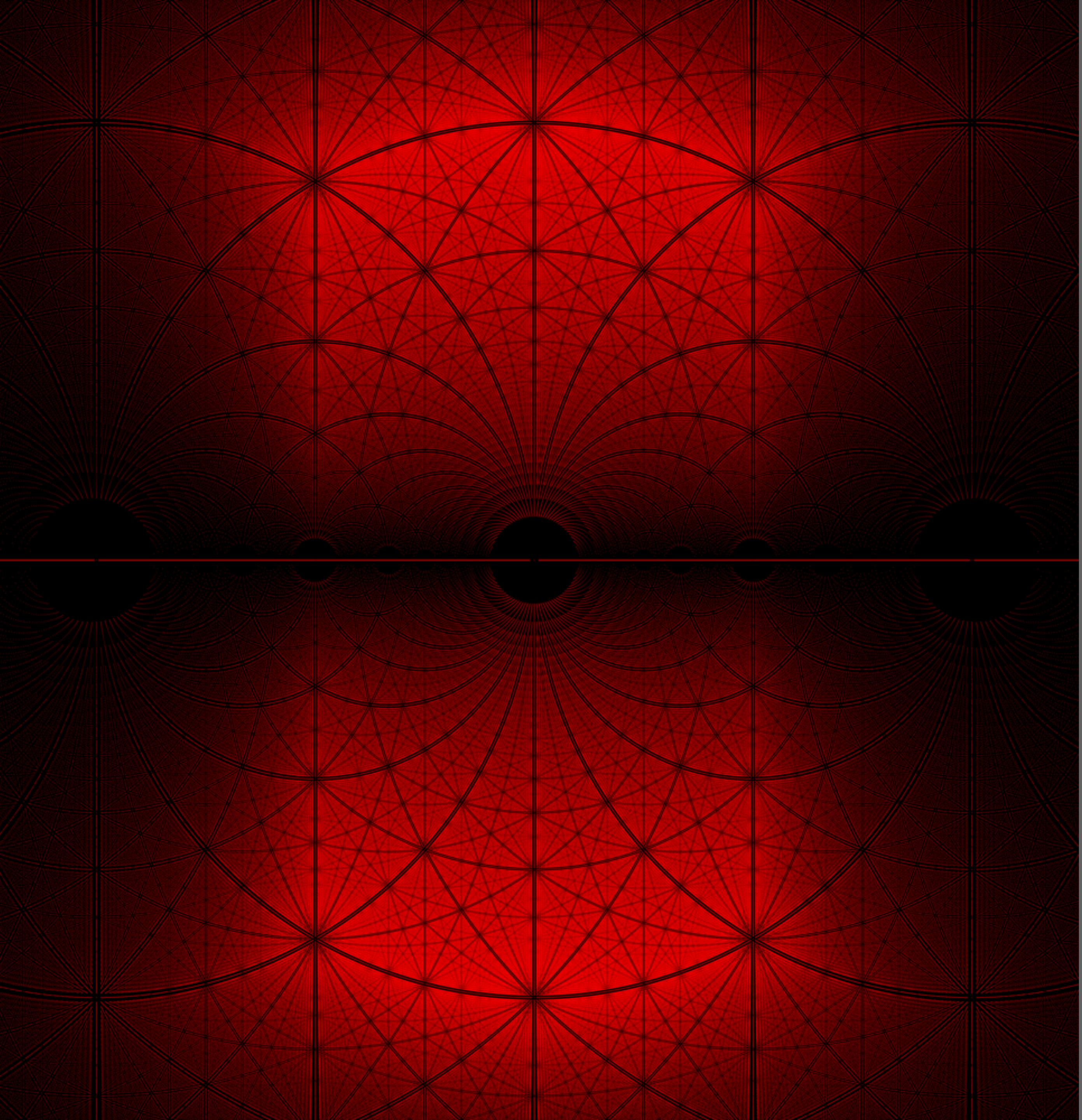
Why Are There Repeats?

- A and A^T have the same eigenvalues
- A and PAP^{-1} have the same eigenvalues
  - If P is a permutation matrix and A is Bohemian, then PAP^{-1} is too. ($n!$ such symmetries)
- Number of coefficients of characteristic polynomials $= n$, not n^2. Therefore, $A \to \mathrm{charpoly}(A)$ is a compression; matrices with same $\mathrm{trace}(A)$, $\mathrm{trace}(A^2), \ldots$ (Fadeev/Leverrier) have same characteristic polynomial. These are non-linear correlations



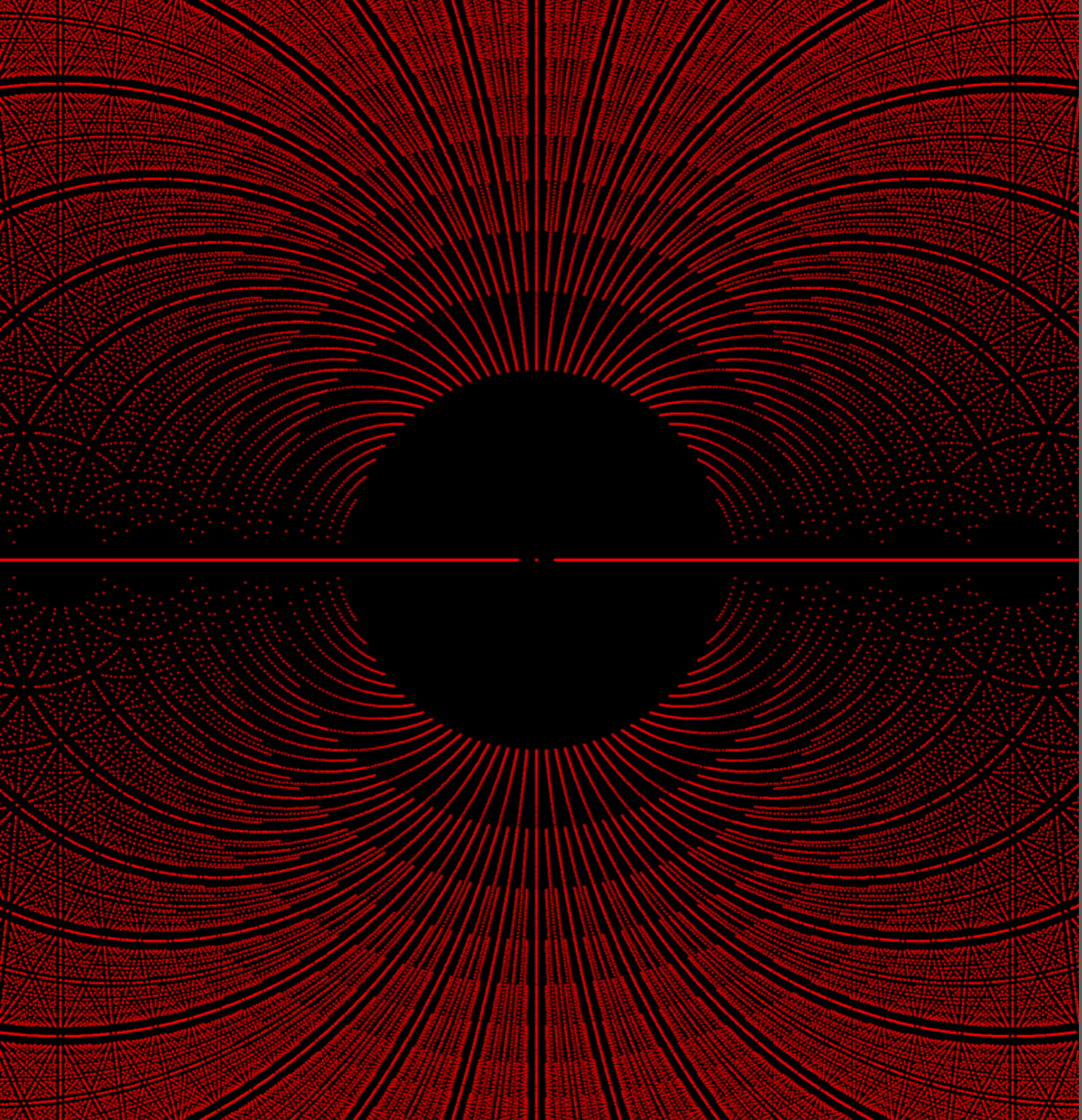
Details

- 5×5 matrices
- Entries in the set $\{-1, 0, 1\}$
- Viewed on $[-0.4 - 0.4i, 0.4 + 0.4i]$



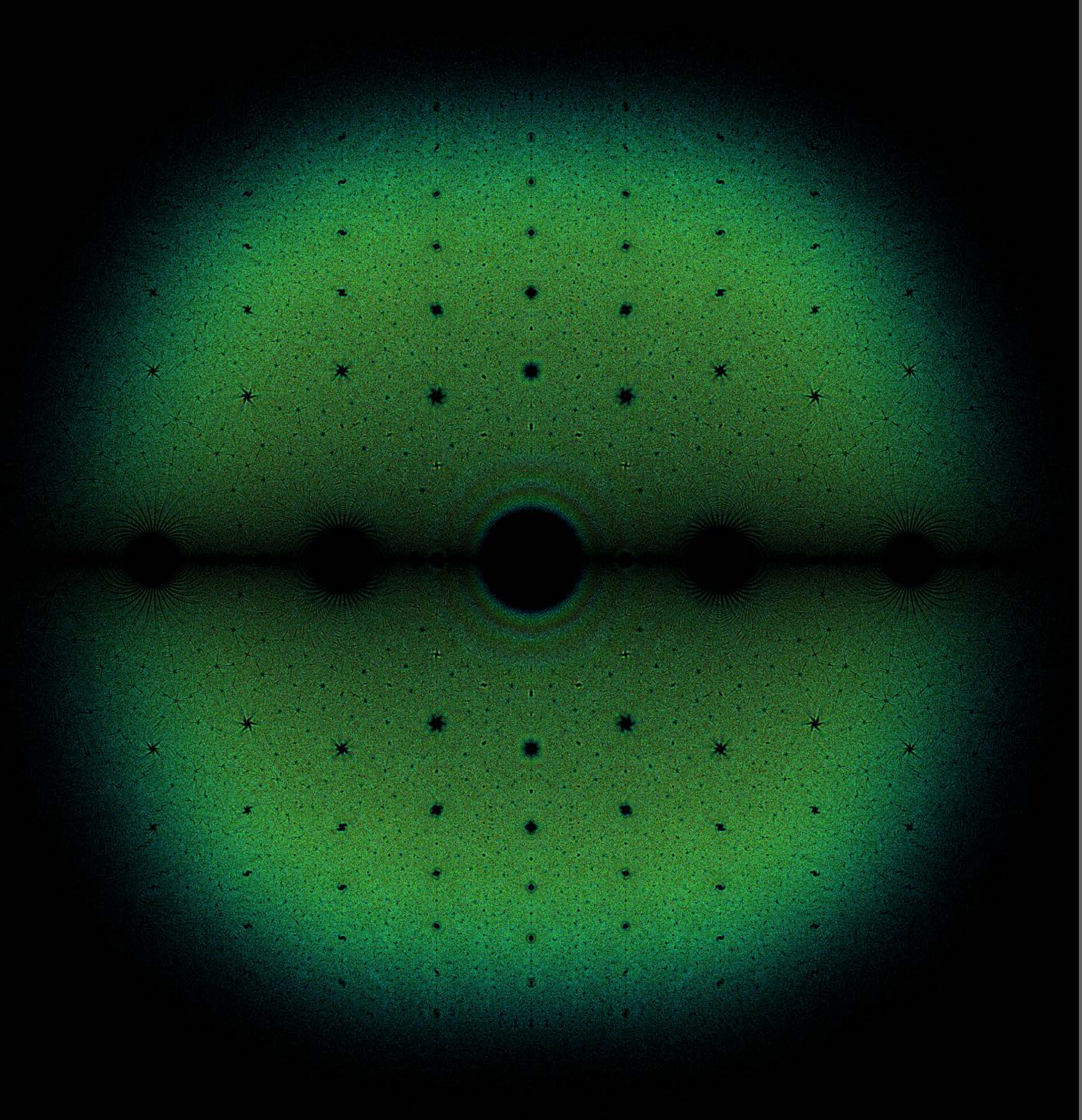
Details

- Algebraic numbers
- Solutions to quadratic polynomials with coefficients no greater than 100 in magnitude
- Viewed on $\pm 1.2 \pm 1.2i$



Details

- Algebraic numbers
- Solutions to quadratic polynomials with coefficients no greater than 100 in magnitude
- Close up of the origin



Details

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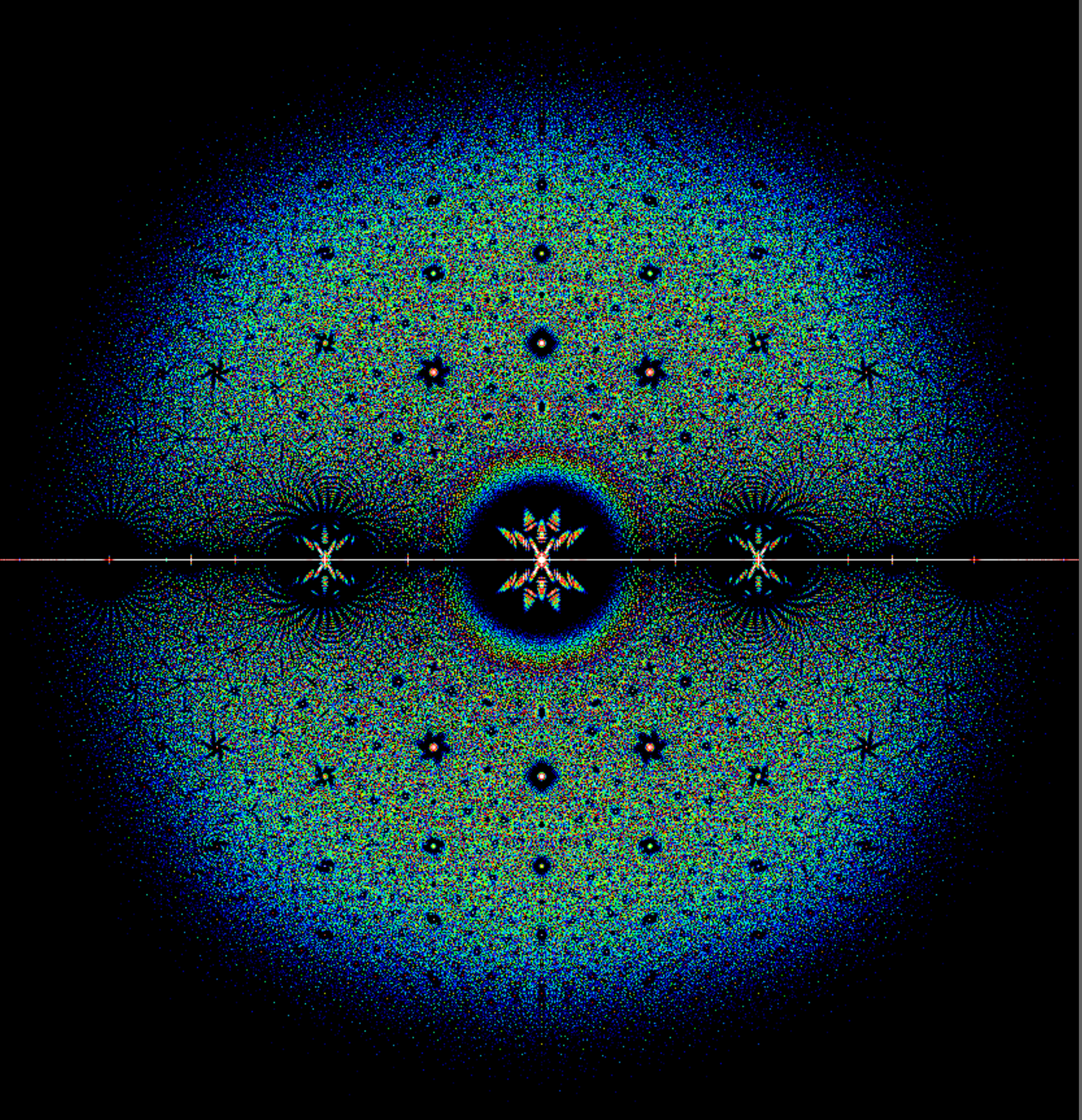
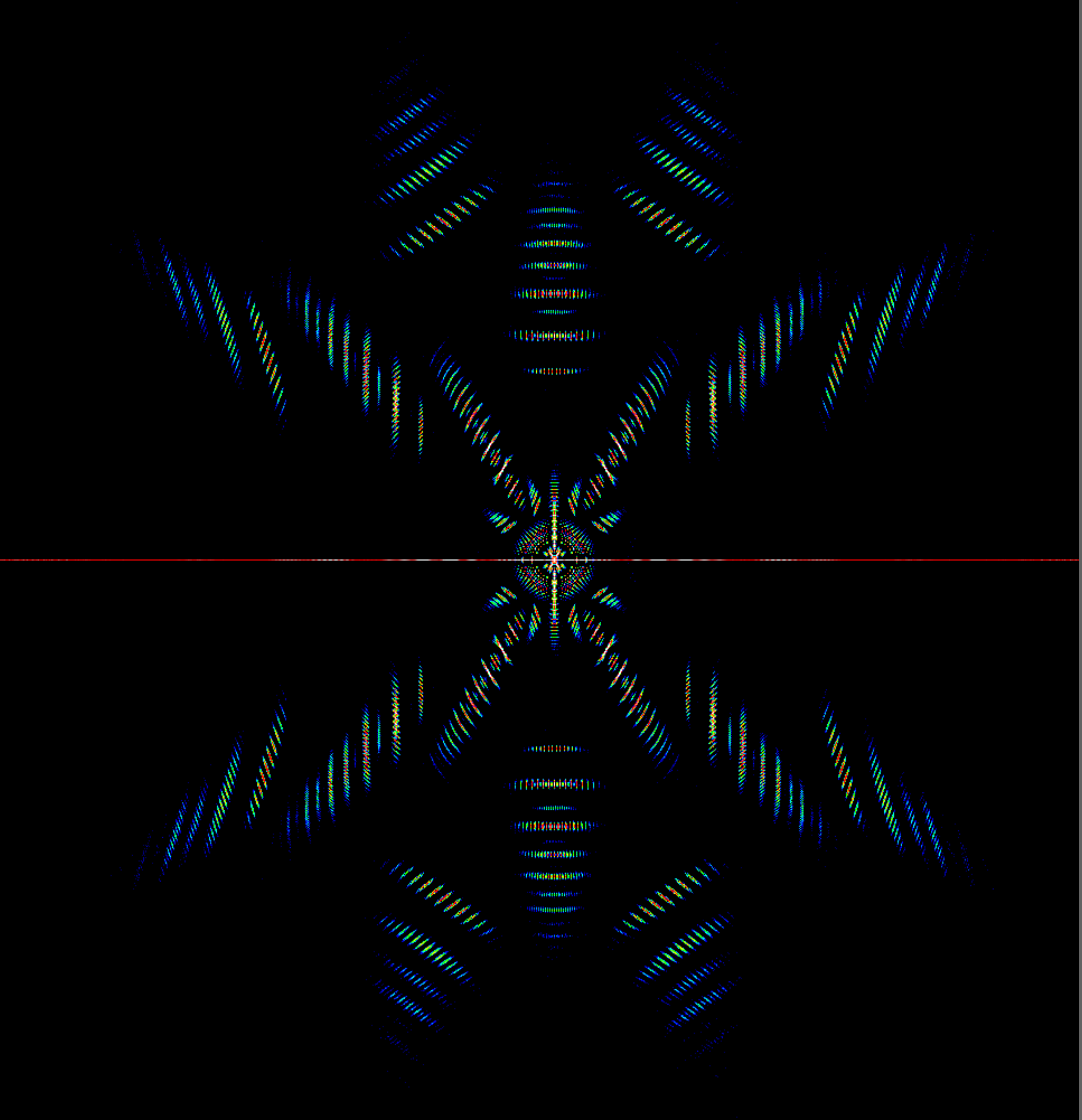
Sampling

- Get the most probable characteristic polynomimals
- Most probable eigenvalues
- Can draw entries from plausible distributions
- Most of the images we've created were done this way



Details

- 5×5 matrices
- Entries sampled uniformly from $\{-1, -1/10000, 0, 1/10000, 1\}$
- Random sample of 50 million matrices



Details

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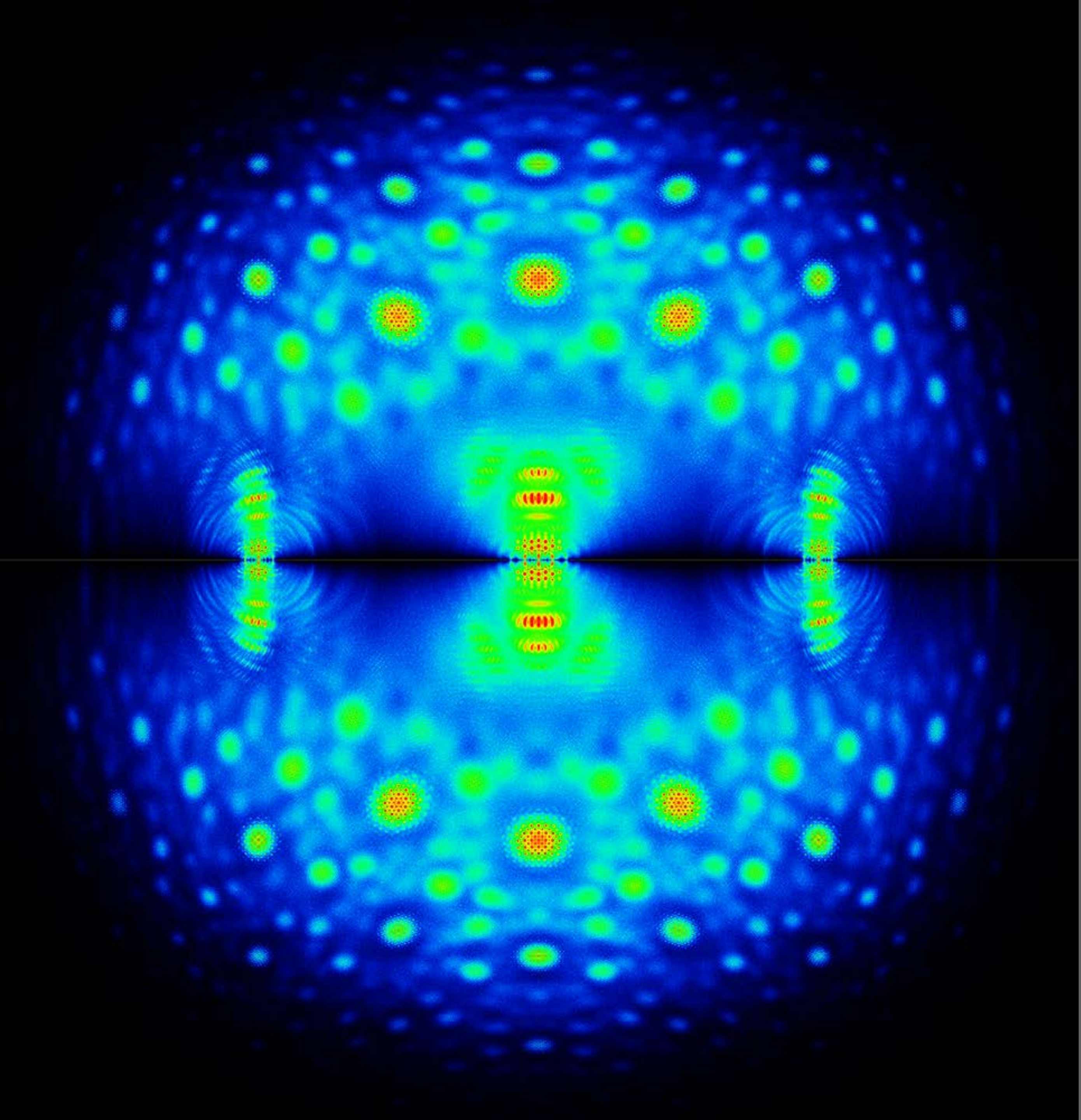
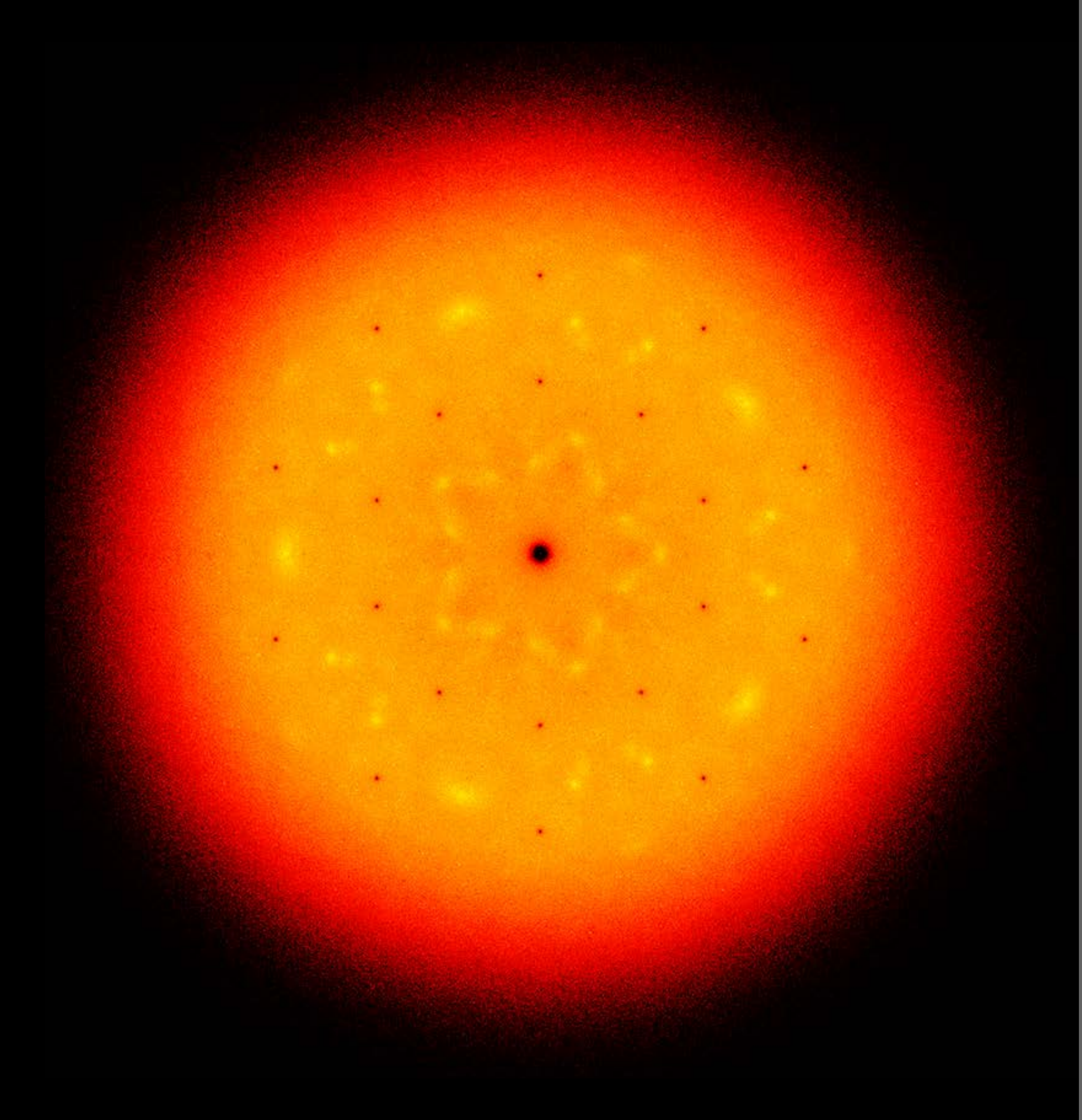


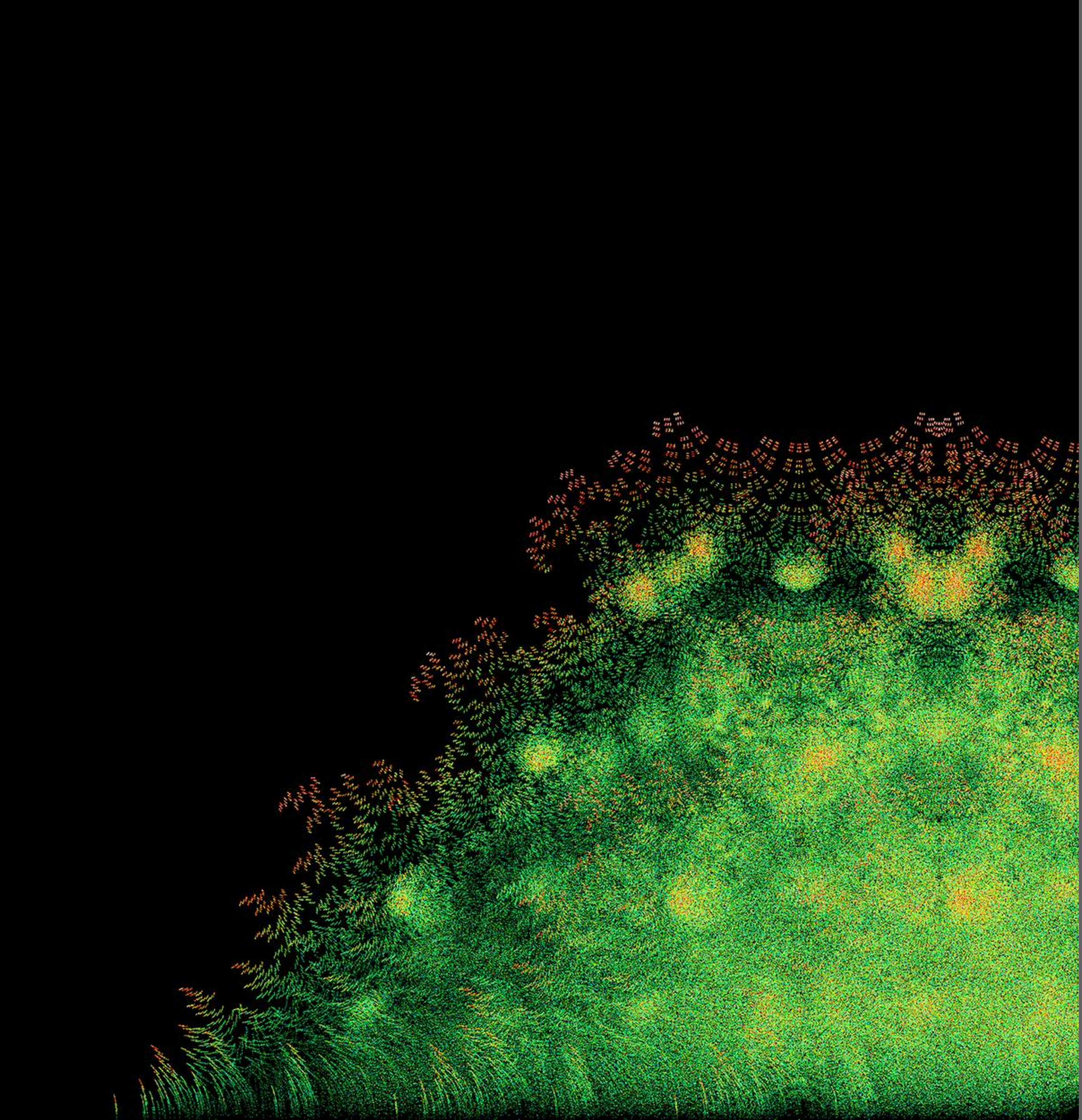
Details

- 5×5 matrices
- Entries sampled uniformly from $\{-20, -1, 0, 1, 20\}$
- Random sample of 73 million matrices
- Colored by density of eigenvalues



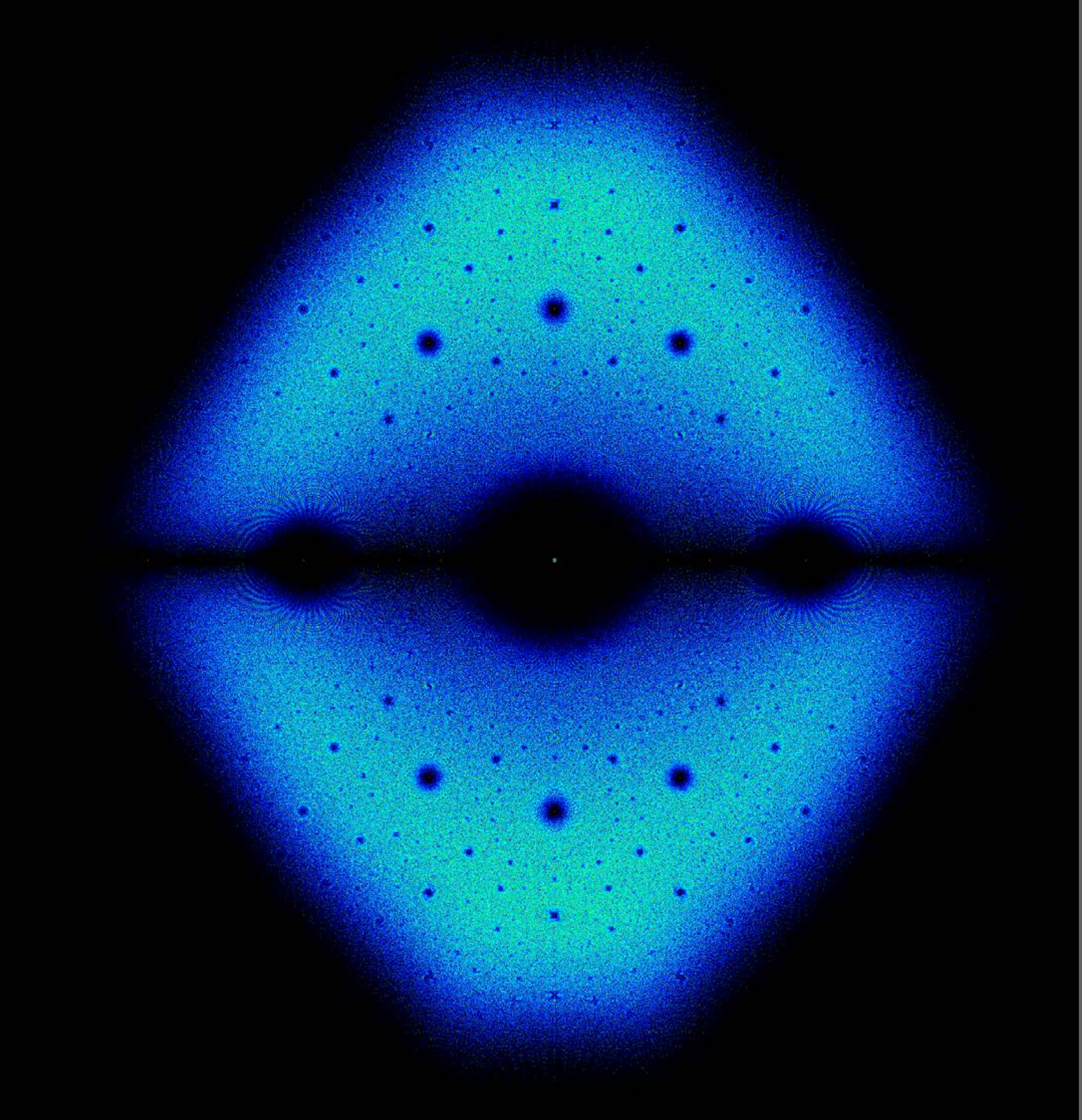
Details

- 5×5 complex symmetric matrices
- Entries sampled uniformly from $\{e^{2\pi i j/5} \mid 0 \leq j < 5\}$
- Random sample of 20 million matrices
- Colored by density of eigenvalues



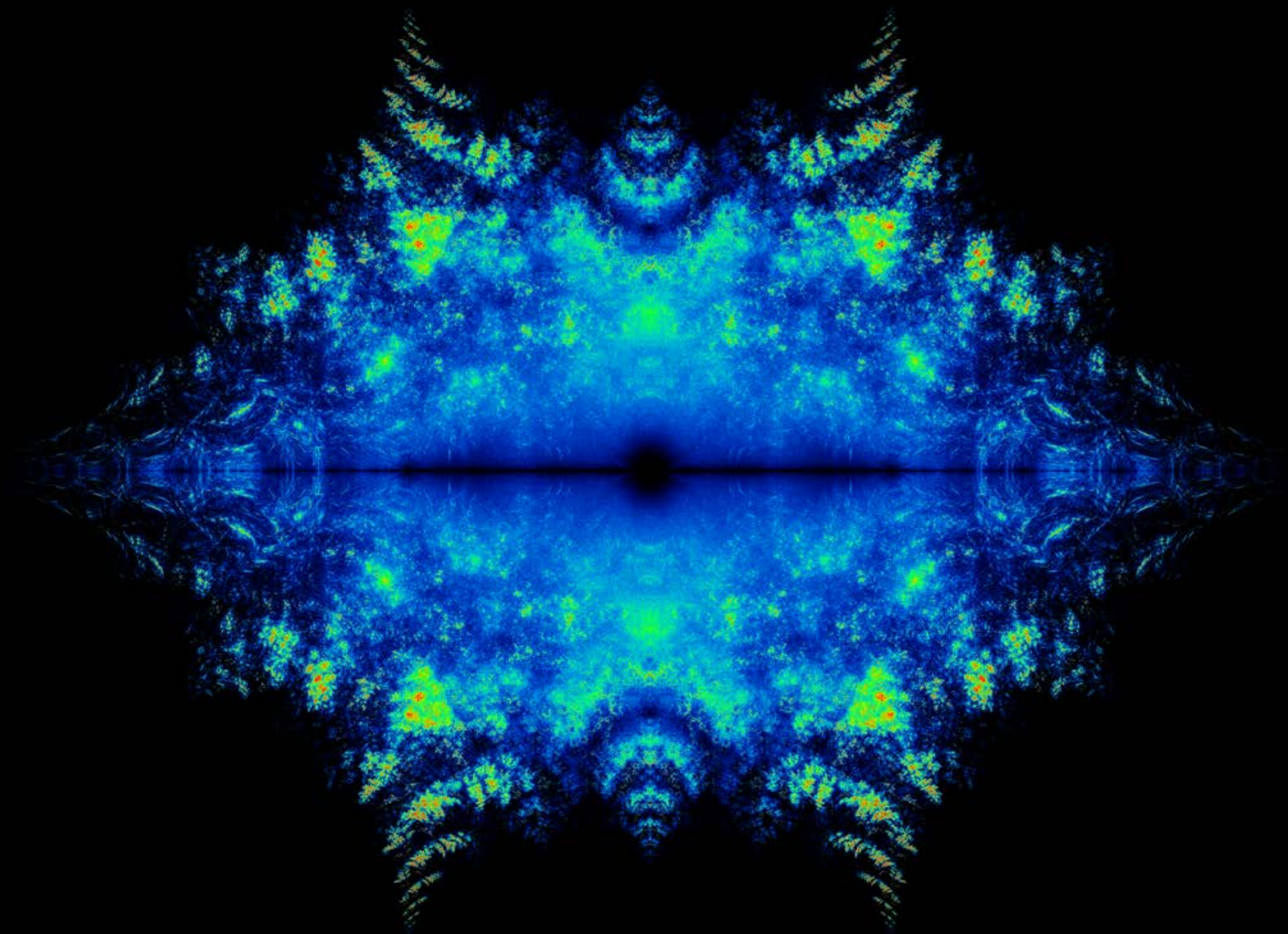
Details

- 12×12 upper Hessenberg Toeplitz matrices
- Entries sampled from $\{-1, 0, 1\}$
- Sample of 10 million matrices
- Viewed on $[-4, 4i]$



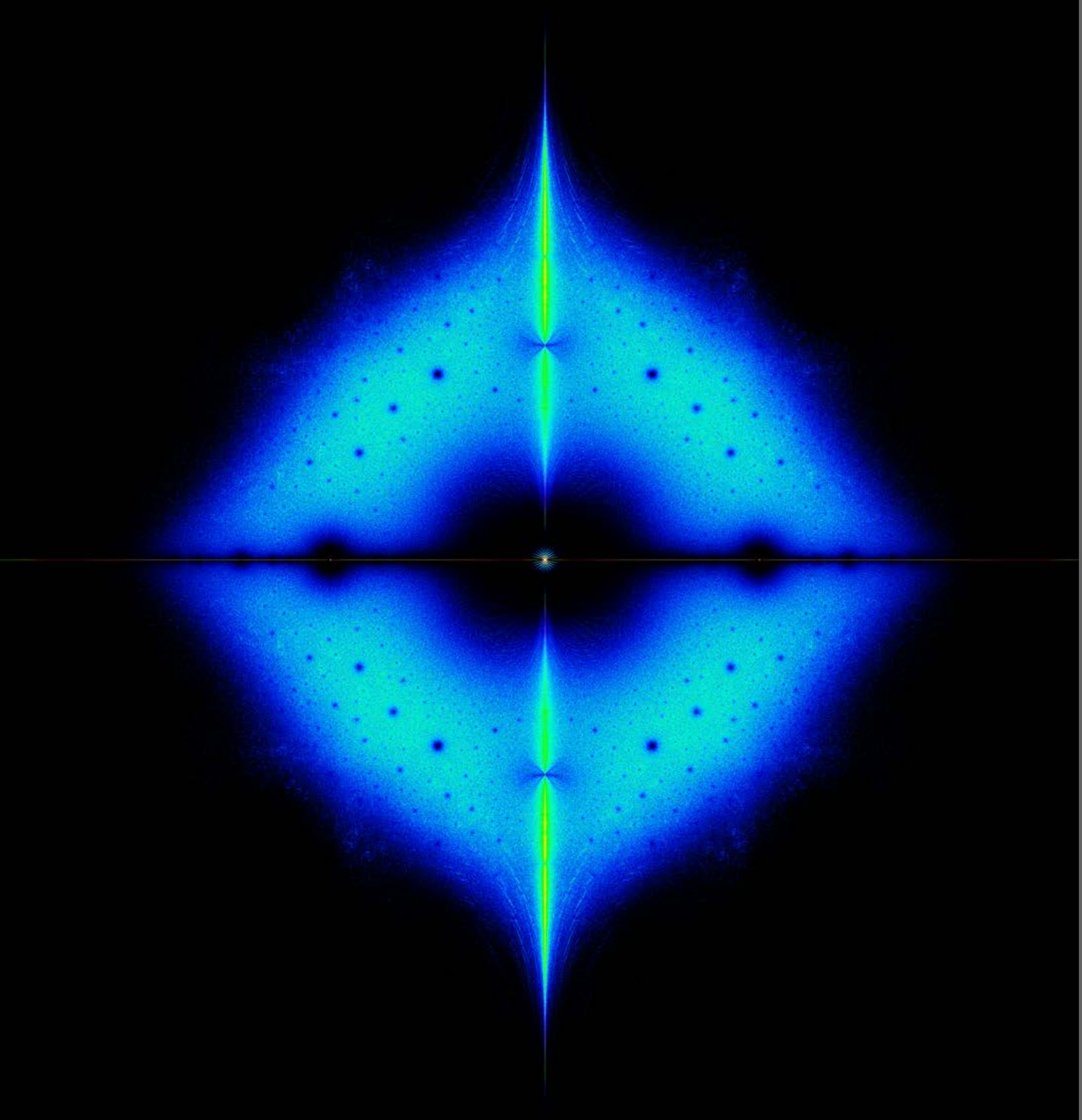
Details

- 12×12 upper Hessenberg Toeplitz matrices with zero main diagonal
- Entries sampled from $\{-1, 0, 1\}$
- Sample of 10 million matrices
- Viewed on $[-2.5-2.5i, 2.5+2.5i]$



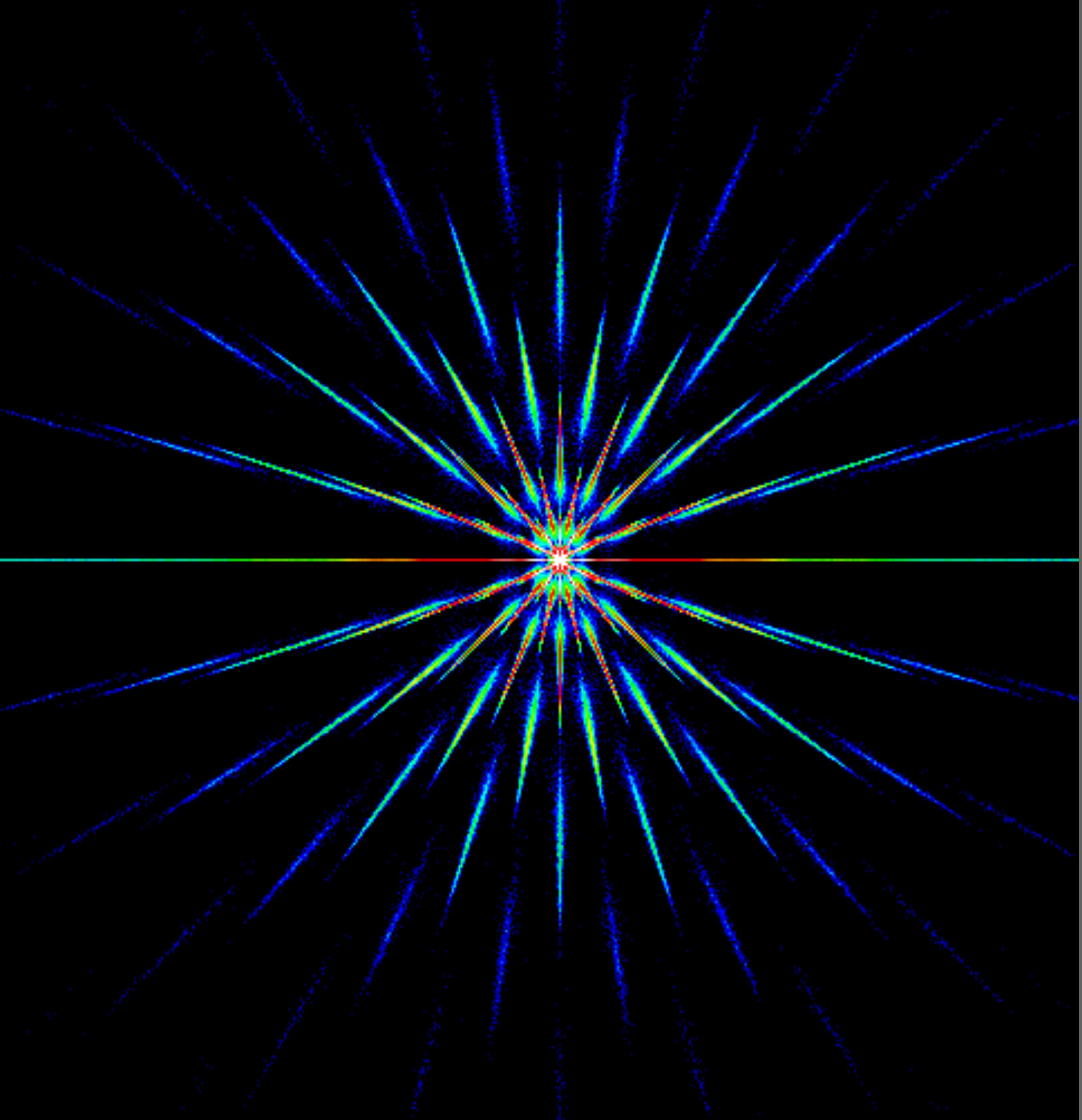
Details

- 20×20 tridiagonal matrices
- Entries sampled uniformly from $\{-1, 1\}$
- Random sample of 25 million matrices
- Colored by density
- Viewed on $[-3-2i, 3+2i]$



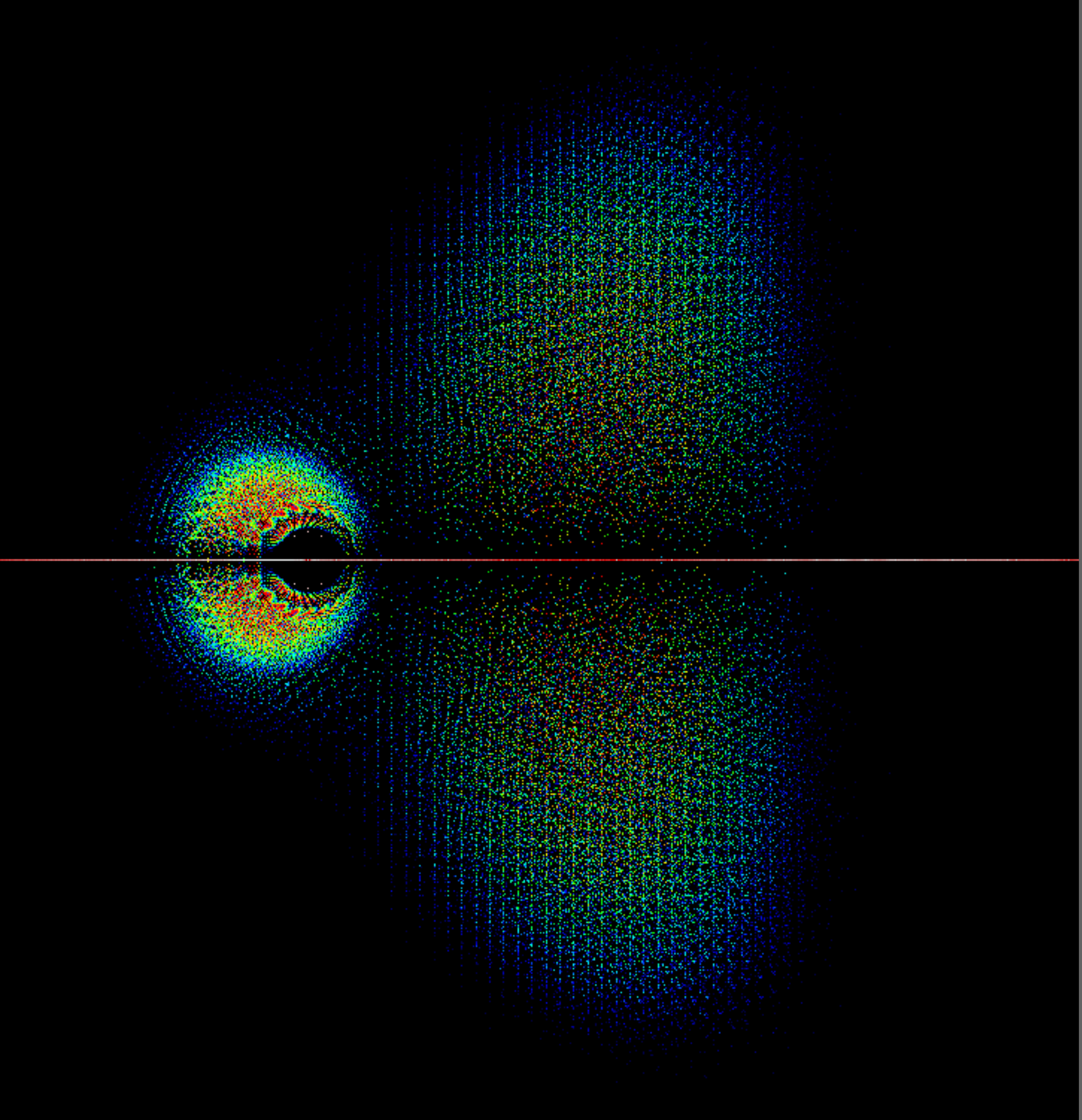
Details

- 20×20 anti-tridiagonal matrices
- Entries sampled uniformly from $\{-1, 0, 1\}$
- Random sample of 25 million matrices
- Colored by density
- Viewed on $[-3-3i, 3+3i]$



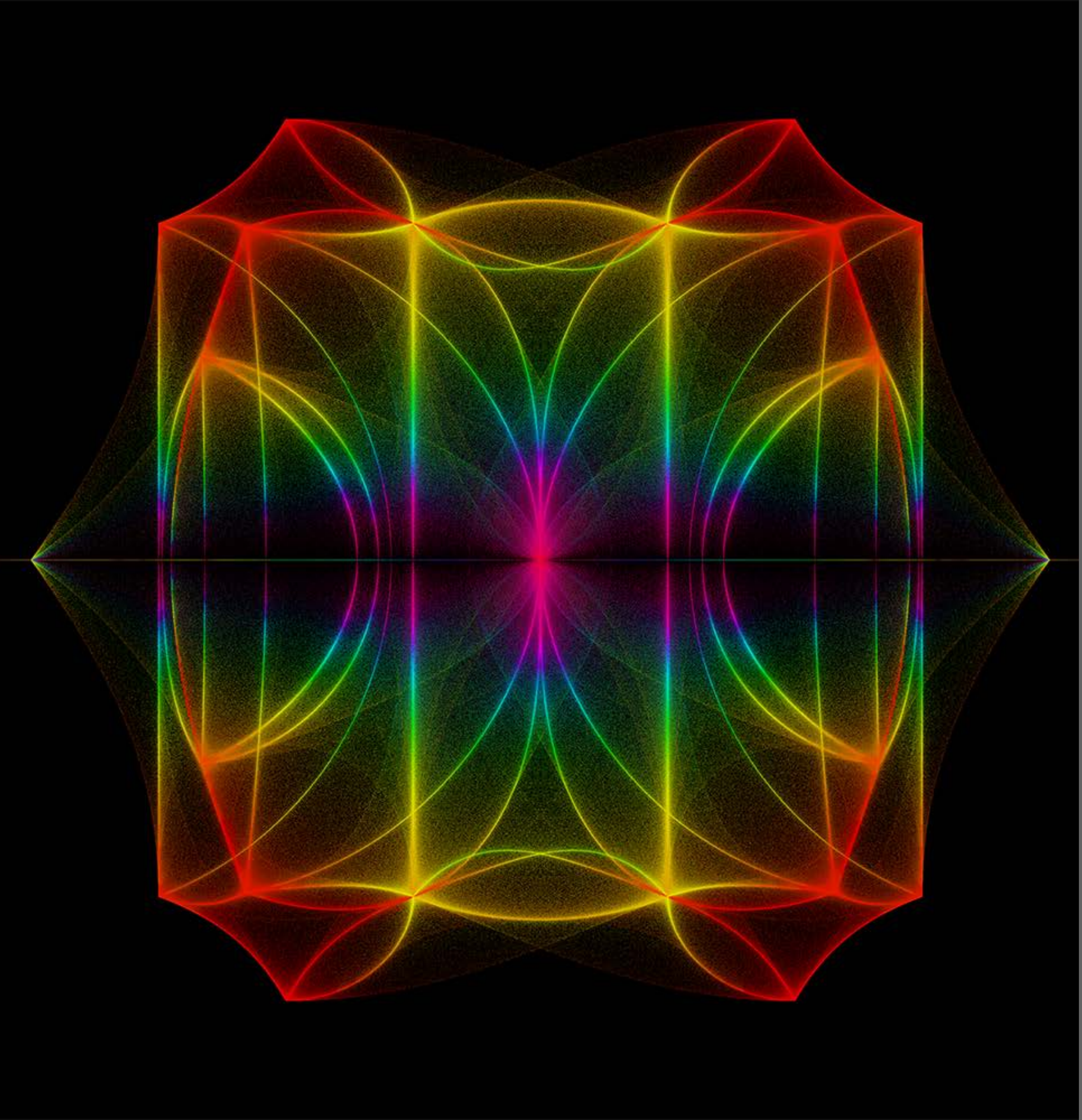
Details

- 20×20 anti-tridiagonal matrices
- Entries sampled uniformly from $\{-1, 0, 1\}$
- Random sample of 25 million matrices
- Colored by density
- Viewed on $[-1/20 - i/20, 1/20 + i/20]$



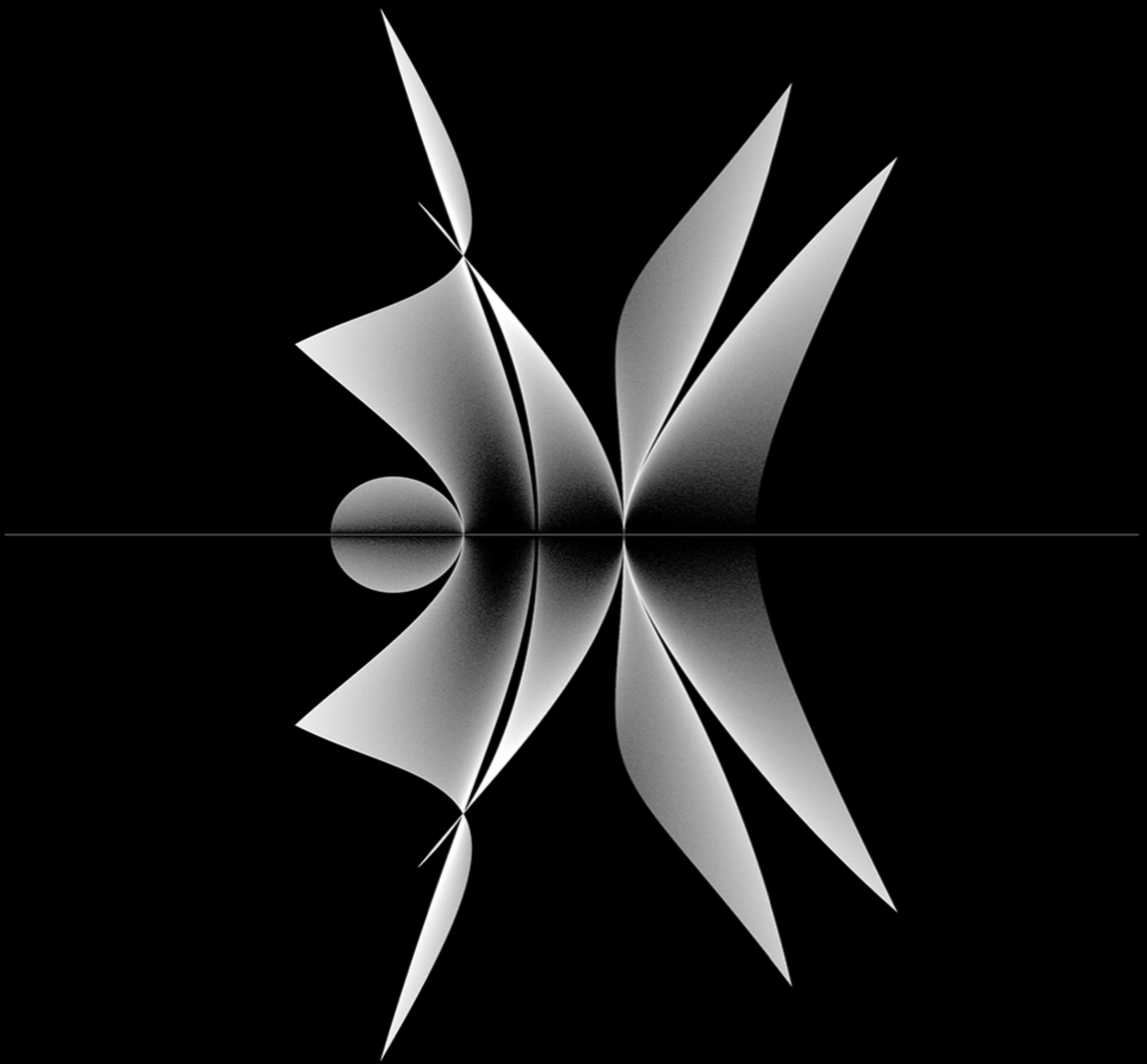
Details

- 60×60 Vandermonde matrices
- Basis entries sampled from $\{-1, 0, 1\}$
- Sample of 10 million matrices
- Viewed on $[-10 - 20i, 30 + 20i]$



Details

- 3×3 matrices
- Each entry is sampled from $2X - 1$ where $X \sim \text{Beta}(0.01, 0.01)$
- Random sample of 10 million matrices
- Colored by eigenvalue condition number

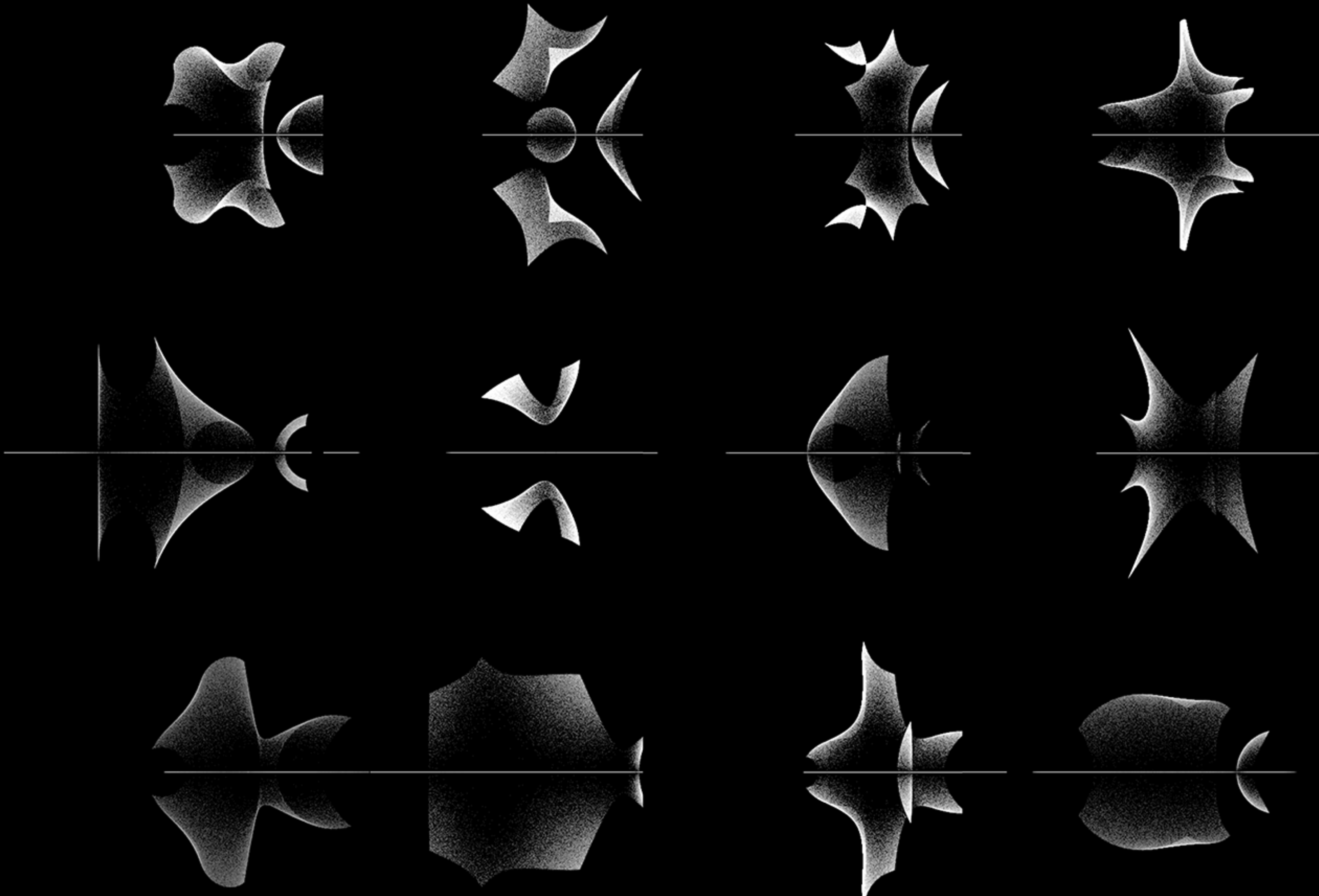


Details

- Eigenvalues of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & A \\ -1 & -1 & 1 & 0 \\ B & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

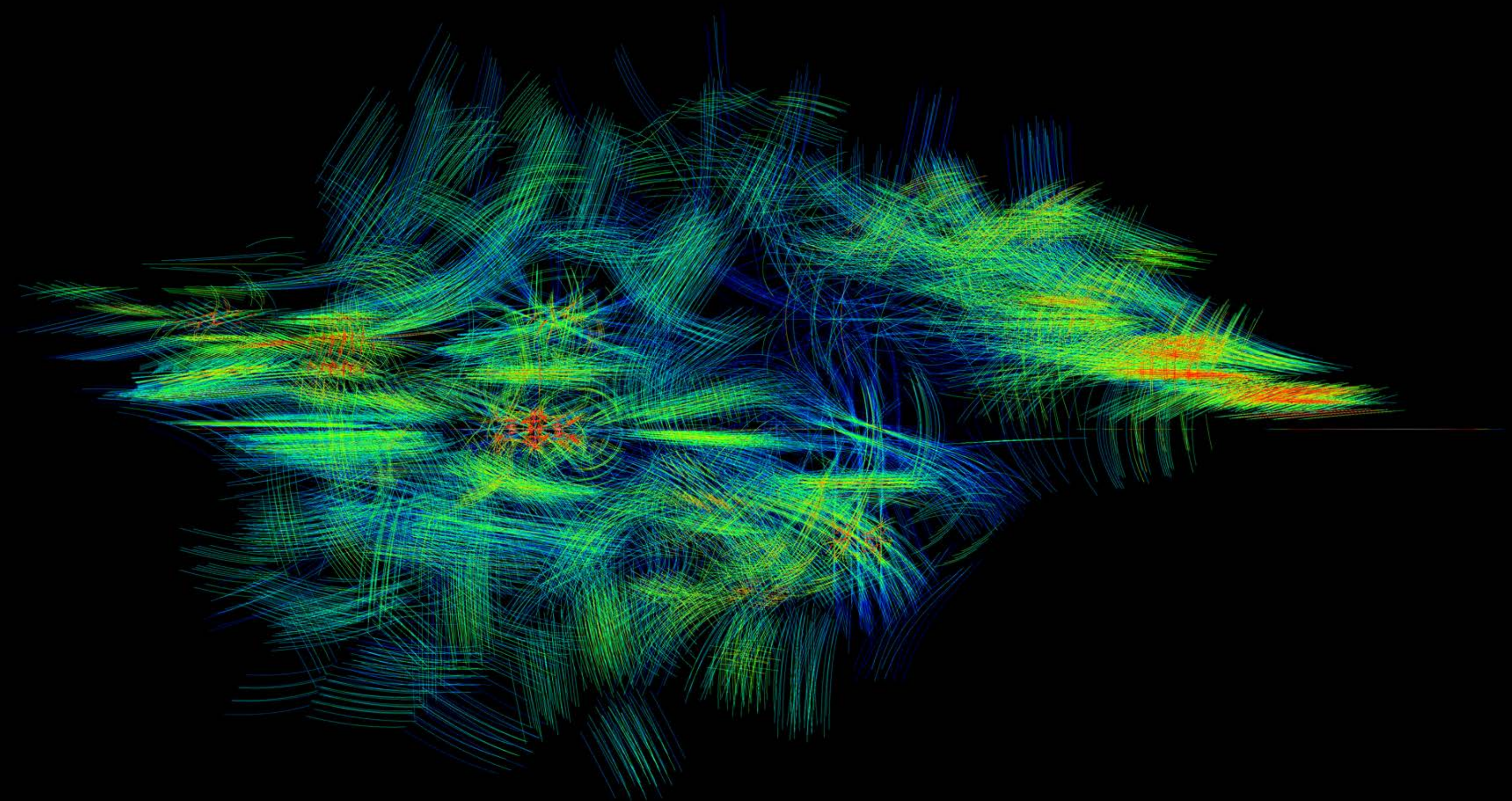
  where A and B are continuous uniform random variables on $(-5, 5)$
- Random sample of 30 million matrices



Future Work

- Analysis of exclusion zones (holes) related to approximation by algebraic numbers
- Analysis of exclusion zones and spectral lines (complex symmetric case)
- Diffraction patterns: wave-like behaviour from particle-like zeros: connection to algebraic approximation

Thank You!



Source code with examples available at:
`github.com/steventhornton/BHIME-Project`

Coming Soon: `BohemianEigenvalues.com`